

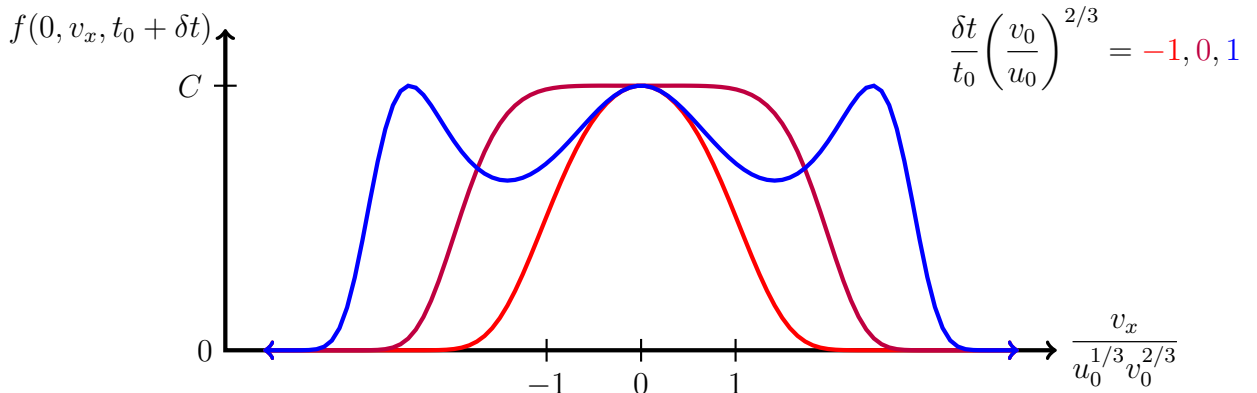
Problem 2

Let the bulk velocity be $-v_0 \sin(kx)\hat{x}$, so the gas converges at $x = 0$. Define $u_0 = \sqrt{\frac{k_B T}{m}}$.

Let $f(x, v_x, t)$ be the single-particle probability density (with periodic boundary conditions to avoid normalization issues). First, let's assume that v_0 is large enough to ignore collisions. Then, f evolves solely through advection, giving $f(x, v_x, t) = f(x - v_x t, v_x, 0)$; we have $f(x, v_x, 0) = C \exp\left(-\frac{1}{2u_0^2}(v_x + v_0 \sin(kx))^2\right)$. Consider the point $x = 0$. Since the nearby gas converges here at time $t_0 = \frac{1}{kv_0}$, we expect non-Maxwellian phenomena to occur here first, if they occur at all. Indeed, for $v_x \ll v_0$ and $\delta t \ll t_0$, note that:

$$\begin{aligned} f(0, v_x, t_0 + \delta t) &= C \exp\left(-\frac{1}{2u_0^2}\left(v_x - v_0 \sin\left(\frac{v_x}{v_0} + kv_x \delta t\right)\right)^2\right) \\ &\approx C \exp\left(-\frac{1}{2u_0^2}\left(v_x - v_0\left(\left(\frac{v_x}{v_0} + kv_x \delta t\right) - \frac{1}{6}\left(\frac{v_x}{v_0}\right)^3\right)\right)^2\right) \\ &= C \exp\left(-\frac{v_x^2}{2u_0^2}\left(\frac{1}{6}\left(\frac{v_x}{v_0}\right)^2 - \frac{\delta t}{t_0}\right)^2\right) \end{aligned}$$

Thus, for $\delta t < 0$, $f(0, v_x, t_0 + \delta t)$ has a single local maximum at $v_x = 0$, and for $\delta t > 0$, it has three local maxima at $v_x = 0, \pm v_0 \sqrt{6 \frac{\delta t}{t_0}}$; its value is C at each of these points. Furthermore, for $\delta t > 0$ there are two local minima at $v_x = \pm v_0 \sqrt{2 \frac{\delta t}{t_0}}$ with value $C \exp\left(-\frac{4}{9}\left(\frac{v_0}{u_0}\right)^2 \left(\frac{\delta t}{t_0}\right)^3\right)$. This shows that multi-stream flow first appears when $\frac{\delta t}{t_0} \sim \left(\frac{u_0}{v_0}\right)^{2/3} \ll 1$. Here, the density is non-negligible for $v_x \lesssim u_0^{1/3} v_0^{2/3} \ll v_0$, so our conditions on v_x and δt are justified.



Now, we'll consider the effect of collisions. Let $f_0(x, v_x, t)$ be the Maxwell-Boltzmann distribution with the same mean and variance in v_x as $f(x, v_x, t)$; this is the local equilibrium distribution. In fact, for spatially homogeneous f close to f_0 , the relaxation time approximation to the Boltzmann equation states that $f - f_0$ will exponentially decay with time scale τ , which is on the order of the mean free time. We have $\tau \sim \frac{1}{n(x, t)u(x, t)r^2}$, where $n(x, t)$ is the local number density and $u(x, t)$ is the local uncertainty in speed. For $\frac{\delta t}{t_0} \lesssim \left(\frac{u_0}{v_0}\right)^{2/3}$, the above result for $f(0, v_x, t)$ gives:

$$n(0, t_0 + \delta t) = n \frac{\int_{-\infty}^{\infty} f(0, v_x, t + \delta t) dv_x}{\int_{-\infty}^{\infty} f(0, v_x, 0) dv_x} \sim n \frac{C u_0^{1/3} v_0^{2/3}}{C u_0} \sim n \left(\frac{v_0}{u_0}\right)^{2/3}$$

Furthermore, $u(0, t_0 + \delta t) \sim u_0^{1/3} v_0^{2/3}$ is the width scale of the distribution. Thus, $\tau \sim \frac{u_0^{1/3}}{v_0^{4/3} n r^2}$. On the other hand, we see that near time t_0 , the size of $f - f_0$ grows with time scale $\tau' = \left(\frac{u_0}{v_0}\right)^{2/3} t_0$ (the same time scale for the shape of f to qualitatively change from Gaussian to multi-stream). For the growth of the multi-stream perturbation to overwhelm the effect of relaxation, it must be true that $\tau' \lesssim \tau$, or $v_0 \gtrsim u_0 \left(\frac{n r^2}{k}\right)^3$.