

As follows from the problem statement, both planes exert a frictional force on the brick, which depends only on the direction of the relative velocity between the brick and the plane. With respect to the upper plane, the brick moves with the sum of the velocity $-\vec{v} = -(v, 0)$ (along the x -axis) and the velocity $\vec{u} = (u_x, u_y)$; the resulting relative velocity is therefore $\vec{u} - \vec{v}$. The frictional acceleration opposing this motion does not depend on the magnitude of the velocity, hence $\vec{a}_1 \sim -\frac{\vec{u} - \vec{v}}{|\vec{u} - \vec{v}|}$. Denoting the proportionality constant by K , we can write $\vec{a}_1 = -K \frac{\vec{u} - \vec{v}}{|\vec{u} - \vec{v}|}$.

With respect to the lower plane, the brick moves just with velocity $\vec{u}$, so the acceleration due to friction of this plane is $\vec{a}_2 = -K \frac{\vec{u}}{|\vec{u}|}$. The resulting acceleration of the brick in the reference frame of

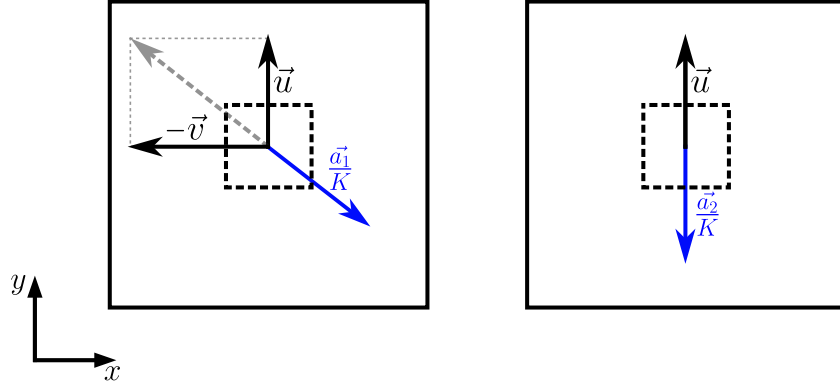


Figure 1: Projection of the relative velocity of the brick with respect to the upper plane (**left**) and the lower plane (**right**).

the lower plane is therefore $\vec{a}_1 + \vec{a}_2$:

$$\frac{d\vec{u}}{dt} = \vec{a}_1 + \vec{a}_2 = -K \left(\frac{\vec{u} - \vec{v}}{|\vec{u} - \vec{v}|} + \frac{\vec{u}}{|\vec{u}|} \right). \quad (1)$$

The value of the constant K can be determined from the known initial acceleration magnitude $|\vec{a}(0)| = a_0$, which gives

$$K = \frac{a_0}{\sqrt{2 + \frac{2u_0}{\sqrt{u_0^2 + v^2}}}}. \quad (2)$$

We can now notice the relationship between (1) and the expression $|\vec{u} - \vec{v}| + |\vec{u}|$, since $\vec{v}$ is constant in time and thus $\frac{d}{dt}(|\vec{u} - \vec{v}| + |\vec{u}|) = \left(\frac{\vec{u} - \vec{v}}{|\vec{u} - \vec{v}|} + \frac{\vec{u}}{|\vec{u}|} \right) \cdot \frac{d\vec{u}}{dt}$. This suggests introducing a new quantity $I = |\vec{u} - \vec{v}| - |\vec{u}|$, for which the time derivative satisfies

$$\frac{dI}{dt} = \left(\frac{\vec{u} - \vec{v}}{|\vec{u} - \vec{v}|} - \frac{\vec{u}}{|\vec{u}|} \right) \cdot \frac{d\vec{u}}{dt} = -K \left(\frac{\vec{u} - \vec{v}}{|\vec{u} - \vec{v}|} - \frac{\vec{u}}{|\vec{u}|} \right) \cdot \left(\frac{\vec{u} - \vec{v}}{|\vec{u} - \vec{v}|} + \frac{\vec{u}}{|\vec{u}|} \right) = -K \left(\frac{(\vec{u} - \vec{v})^2}{|\vec{u} - \vec{v}|^2} - \frac{\vec{u}^2}{|\vec{u}|^2} \right) = 0. \quad (3)$$

This shows that I is time invariant, i.e. a constant whose value is $I(t) = I(0) = \sqrt{u_0^2 + v^2} - u_0$. If we denote the instantaneous magnitude of $\vec{u}$ as $u = |\vec{u}| = \sqrt{u_x^2 + u_y^2}$, then $I = \sqrt{u^2 + v^2} - 2u_x v - u$. Squaring this formula gives us

$$\begin{aligned} (I + u)^2 &= u^2 + v^2 - 2u_x v \\ 2u_x v &= u^2 + v^2 - I^2 - u^2 - 2Iu \\ u_x &= \frac{v^2 - I^2 - 2Iu}{2v} = I \frac{u_0 - u}{v}, \end{aligned} \quad (4)$$

where in the last step we used the relation $v^2 - I^2 = 2Iu_0$. From (4) we can also derive an implicit expression for the u_y component:

$$u_y(t) = \sqrt{u^2(t) - u_x^2(t)} = \sqrt{u^2(t) - \frac{I^2}{v^2}(u_0 - u(t))^2}. \quad (5)$$

The desired displacement l_y can then be obtained as the time integral of (5):

$$l_y = \int_0^\infty u_y dt = \int_0^\infty \sqrt{u^2(t) - \frac{I^2}{v^2}(u_0 - u(t))^2} dt. \quad (6)$$

Since we do not know the explicit time dependence $u(t)$, we need to transform the integral using a suitable substitution. For that, we use the y -component of equation (1) and then the time derivative of the expression (5):

$$\frac{du_y}{dt} = -K \left(\frac{u_y}{|\vec{u} - \vec{v}|} + \frac{u_y}{u} \right) = -Ku_y \frac{2u + I}{u(u + I)} \implies dt = -\frac{u(u + I)}{K(2u + I)u_y} du_y, \quad (7)$$

$$\frac{du_y}{du} = \frac{u + \frac{I^2}{v^2}(u_0 - u)}{u_y} \implies du_y = \frac{u + \frac{I^2}{v^2}(u_0 - u)}{\sqrt{u^2 - \frac{I^2}{v^2}(u_0 - u)^2}} du. \quad (8)$$

We also need to determine the limits of integration in the variable u . The initial velocity is $u(0) = u_0$. The final velocity must clearly be aligned with the x -axis, i.e. $u(\infty) = u_x(\infty)$; substituting into (4) gives

$$\begin{aligned} u(\infty) = u_\infty &= I \frac{u_0 - u_\infty}{v} \\ u_\infty &= \frac{u_0}{1 + \frac{v}{I}} = \frac{v - I}{2}. \end{aligned} \quad (9)$$

Substituting (7) and (8) into (6), we obtain the final analytically solvable definite integral:

$$\begin{aligned} l_y &= -\frac{u_0 I}{K v^2} \int_{u_0}^{u_\infty} \frac{u(u + I)}{\sqrt{u^2 - \frac{I^2}{v^2}(u_0 - u)^2}} du = \dots \\ &= \frac{1}{8a_0} \sqrt{\frac{2u_0}{\sqrt{u_0^2 + v^2}}} \left(\sqrt{u_0(2u_0 + I)^3} + \frac{4Iu_0 - v^2}{\sqrt{2}} \ln \left(\frac{\sqrt{2u_0(2u_0 + I)} + 2u_0 + I}{v} \right) \right). \end{aligned} \quad (10)$$

By putting in the values $u_0 = v = 1 \text{ m s}^{-1}$ and $a_0 = 1 \text{ m s}^{-2}$ into (10), we get the final result:

$$l_y = 8^{-\frac{5}{4}} \left(2\sqrt{1 + \sqrt{2}} + 2\sqrt{2(1 + \sqrt{2})} + (8 - 5\sqrt{2}) \ln \left(1 + \sqrt{2} + \sqrt{2(1 + \sqrt{2})} \right) \right) \text{ m} \doteq \boxed{0.6631 \text{ m}}. \quad (11)$$