

Brick between planes

Thach Nguyen

October 23, 2025

Reference Solution

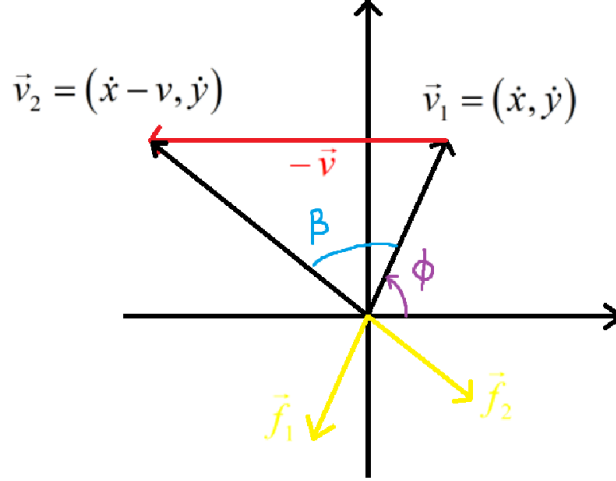


Figure 1: Relative velocities and their corresponding friction forces $\vec{f}_1$ and $\vec{f}_2$ ($f_1 = f_2 = f$).

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the velocity of the brick with respect to the lower and higher planes (Fig. 1). Then, we have

$$\mathbf{v}_1 = (\dot{x}, \dot{y}), \quad \mathbf{v}_2 = (\dot{x} - v, \dot{y}),$$

and denote their magnitudes $u = \|\mathbf{v}_1\|$, $w = \|\mathbf{v}_2\|$. The equation of motion is

$$m\dot{\mathbf{v}}_1 = -f \frac{\mathbf{v}_1}{u} - f \frac{\mathbf{v}_2}{w}, \quad (1)$$

with the same right-hand side for $\dot{\mathbf{v}}_2$ because the upper plate's velocity is constant.

Invariant $w - u$

Differentiating $u = \|\mathbf{v}_1\|$ and $w = \|\mathbf{v}_2\|$,

$$\dot{u} = \frac{\mathbf{v}_1 \cdot \dot{\mathbf{v}}_1}{u}, \quad \dot{w} = \frac{\mathbf{v}_2 \cdot \dot{\mathbf{v}}_2}{w}.$$

Using (1) and $\dot{\mathbf{v}}_2 = \dot{\mathbf{v}}_1$,

$$\dot{w} - \dot{u} = \dot{\mathbf{v}}_1 \cdot \left(\frac{\mathbf{v}_2}{w} - \frac{\mathbf{v}_1}{u} \right) = -\frac{f}{m} (\hat{\mathbf{v}}_1 + \hat{\mathbf{v}}_2) \cdot (\hat{\mathbf{v}}_2 - \hat{\mathbf{v}}_1) = 0,$$

hence

$$w - u = C = \text{const}, \quad C = w(0) - u(0) = \sqrt{u_0^2 + v^2} - u_0 > 0. \quad (2)$$

Angle dynamics

Let ϕ be the angle of $\mathbf{v}_1$ from $+\hat{\mathbf{x}}$, so $\tan \phi = \dot{y}/\dot{x}$ and $\sin \phi = \dot{y}/\sqrt{\dot{y}^2 + \dot{x}^2} = \dot{y}/u$. The rotation of $\vec{v}_1$ is due to the projection component $f_2 \sin \beta$. Using (1) together with the law of sines gives

$$u \dot{\phi} = -\frac{f}{m} \sin \beta = -\frac{f}{m} \frac{v \sin \phi}{w}. \quad (3)$$

Moreover, by the cosine law for that same triangle of sides u, w, v and using (2),

$$(u + C)^2 = u^2 + v^2 - 2uv \cos \phi \quad \Rightarrow \quad u(\phi) = \frac{v^2 - C^2}{2(C + v \cos \phi)}, \quad w(\phi) = u(\phi) + C. \quad (4)$$

y -displacement as a parameter integral

From (3),

$$dt = -\frac{m u w}{f v \sin \phi} d\phi, \quad \Rightarrow \quad dy = \dot{y} dt = u \sin \phi dt = -\frac{m}{f v} u^2 w d\phi.$$

The initial direction is $\phi(0) = \pi/2$ and $\phi \rightarrow 0$ as $t \rightarrow \infty$, hence

$$\ell_y = \int_0^\infty \dot{y} dt = \frac{m}{f v} \int_0^{\pi/2} u(\phi)^2 w(\phi) d\phi, \quad (5)$$

with u, w from (4). Setting C to a and v to b for better visuals,

$$u(\phi) = \frac{b^2 - a^2}{2(a + b \cos \phi)}, \quad w(\phi) = u(\phi) + a,$$

The integrand, then, is a linear combination of $(a + b \cos \phi)^{-2}$ and $(a + b \cos \phi)^{-3}$. Define

$$J_n(a, b) = \int_0^{\pi/2} \frac{d\phi}{(a + b \cos \phi)^n} \quad (n \geq 1),$$

for which $J_{n+1} = -\frac{1}{n} \partial J_n / \partial a$. Using the standard

$$J_1(a, b) = \frac{2}{\sqrt{b^2 - a^2}} \operatorname{arctanh} \sqrt{\frac{b-a}{b+a}},$$

one obtains closed forms for J_2, J_3 by differentiating with respect to a and combines them in (5). After simplification the result is

$$\ell_y = \frac{m}{f} \left[\frac{v^2 u_0}{8C} + \frac{\sqrt{2u_0 C}}{8v} (v^2 - 2C^2) \operatorname{arctanh} \sqrt{\frac{v-C}{v+C}} \right], \quad C = \sqrt{u_0^2 + v^2} - u_0.$$

Relating f/m to the given $|\mathbf{a}_0|$

At $t = 0$ we have $\mathbf{v}_1 = (0, u_0)$ and $\mathbf{v}_2 = (-v, u_0)$, hence (from (1))

$$a_{0x} = \frac{f}{m} \frac{v}{\sqrt{u_0^2 + v^2}}, \quad a_{0y} = -\frac{f}{m} \left(1 + \frac{u_0}{\sqrt{u_0^2 + v^2}}\right).$$

Therefore, with $V \equiv \sqrt{u_0^2 + v^2}$,

$$|\mathbf{a}_0| = \frac{f}{m} \sqrt{\left(\frac{v}{V}\right)^2 + \left(1 + \frac{u_0}{V}\right)^2} = \frac{f}{m} \sqrt{2\left(1 + \frac{u_0}{V}\right)}.$$

Solving for f/m gives

$$\boxed{\frac{f}{m} = \frac{|\mathbf{a}_0|}{\sqrt{2\left(1 + \frac{u_0}{\sqrt{u_0^2 + v^2}}\right)}} = \frac{1}{\sqrt{2 + \sqrt{2}}}}, \quad (6)$$

with $v = u_0 = 1$ m/s and $|\mathbf{a}_0| = 1$ m/s². As long as we don't change how hard the plates are pressed or the mass of the brick, we don't have to calculate this ratio again.

Final form of ℓ_y in terms of

Since ℓ_y above appears multiplied by m/f , replace m/f with the reciprocal of (6). With

$$C = \sqrt{u_0^2 + v^2} - u_0, \quad V = \sqrt{u_0^2 + v^2},$$

we obtain

$$\boxed{\ell_y = \left(\sqrt{2 + \sqrt{2}}\right) \left[\frac{v^2 u_0}{8C} + \frac{\sqrt{2} u_0 C}{8v} (v^2 - 2C^2) \operatorname{arctanh} \sqrt{\frac{v - C}{v + C}} \right]}.$$

This expression depends only on the given u_0 and v . When $v = u_0 = 1$ m/s,

$$\boxed{\ell_y = 0.6631 \text{ m.}}$$

Checks. The expression above satisfies the two clean limits (fixing f/m and u_0):

$$v \rightarrow 0^+ : \quad \ell_y \rightarrow \frac{m u_0^2}{4f} = 0.4619 \text{ m} \quad (\text{both plates at rest} \Rightarrow \text{decel. } 2f/m),$$

$$v \gg u_0 : \quad \ell_y \rightarrow \frac{2m u_0^2}{3f} = 1.2318 \text{ m} \quad (\text{upper plate drags in } x, \text{ only lower plate damps } y).$$

Our ℓ_y sits right in between them — fully consistent and beautiful.