

Physics Cup - TalTech2026 - Problem 1

Muntean Rares

Full Solution

Step 1: System setup and kinematics

We consider two parallel planes (normal along z). The lower plane is fixed; the upper plane moves with constant velocity $\mathbf{v} = (v, 0, 0)$ along the x axis. A small brick is squeezed between the planes, with initial velocity

$$\mathbf{u}(0) = \mathbf{u}_0 = (0, u_0, 0), \quad u_0 > 0,$$

so that initially the brick moves purely along $+y$. The motion of the center of mass is constrained between the planes; we thus work entirely in the xy plane.

We denote the mass m , the (equal) normal reactions $N_1 = N_2 = N$, and the dry friction coefficient μ (same at both contacts). The friction forces at the lower and upper plane depend on relative velocities:

$$\begin{aligned} \text{bottom plane: } & \mathbf{u} \quad (\text{relative vel.} = \mathbf{u}), \\ \text{top plane: } & \mathbf{u} - \mathbf{v} \quad (\text{relative vel.} = \mathbf{u} - \mathbf{v}). \end{aligned}$$

The magnitudes are μN , directions oppose the corresponding relative velocities.

Step 2: Forces and vector equation of motion

Therefore,

$$\mathbf{F}_1 = -\mu N \frac{\mathbf{u}}{|\mathbf{u}|}, \quad \mathbf{F}_2 = -\mu N \frac{\mathbf{u} - \mathbf{v}}{|\mathbf{u} - \mathbf{v}|}. \quad (1)$$

The total force is

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = -\mu N \left(\frac{\mathbf{u}}{|\mathbf{u}|} + \frac{\mathbf{u} - \mathbf{v}}{|\mathbf{u} - \mathbf{v}|} \right). \quad (2)$$

Newton's second law gives

$$m \dot{\mathbf{u}} = \mathbf{F} \implies \dot{\mathbf{u}} = -\frac{\mu N}{m} \left(\frac{\mathbf{u}}{|\mathbf{u}|} + \frac{\mathbf{u} - \mathbf{v}}{|\mathbf{u} - \mathbf{v}|} \right). \quad (3)$$

Introduce the constant

$$c \equiv \frac{\mu N}{m} (> 0), \quad (4)$$

The compact form of the equation is

$$\boxed{\dot{\mathbf{u}} = -c \left(\frac{\mathbf{u}}{|\mathbf{u}|} + \frac{\mathbf{u} - \mathbf{v}}{|\mathbf{u} - \mathbf{v}|} \right)}. \quad (5)$$

Step 3: Initial acceleration and determination of c from a_0

At $t = 0$:

$$\mathbf{u}_0 = (0, u_0), \quad |\mathbf{u}_0| = u_0, \quad \mathbf{u}_0 - \mathbf{v} = (-v, u_0), \quad |\mathbf{u}_0 - \mathbf{v}| = \sqrt{u_0^2 + v^2}.$$

Therefore

$$\begin{aligned} \dot{\mathbf{u}}|_{t=0} &= -c \left[\frac{(0, u_0)}{u_0} + \frac{(-v, u_0)}{\sqrt{u_0^2 + v^2}} \right] = -c \left[(0, 1) + \left(-\frac{v}{\sqrt{u_0^2 + v^2}}, \frac{u_0}{\sqrt{u_0^2 + v^2}} \right) \right] \\ &= -c \left(-\frac{v}{\sqrt{u_0^2 + v^2}}, 1 + \frac{u_0}{\sqrt{u_0^2 + v^2}} \right). \end{aligned} \quad (6)$$

Thus components:

$$\dot{u}_x(0) = +c \frac{v}{\sqrt{u_0^2 + v^2}}, \quad \dot{u}_y(0) = -c \left(1 + \frac{u_0}{\sqrt{u_0^2 + v^2}} \right). \quad (7)$$

The initial acceleration magnitude $a_0 = |\dot{\mathbf{u}}(0)|$:

$$\begin{aligned} a_0^2 &= \dot{u}_x(0)^2 + \dot{u}_y(0)^2 = c^2 \left[\frac{v^2}{u_0^2 + v^2} + \left(1 + \frac{u_0}{\sqrt{u_0^2 + v^2}} \right)^2 \right] \\ &= c^2 \left[\frac{v^2}{u_0^2 + v^2} + 1 + \frac{2u_0}{\sqrt{u_0^2 + v^2}} + \frac{u_0^2}{u_0^2 + v^2} \right] = c^2 \left[\frac{v^2 + u_0^2}{u_0^2 + v^2} + \frac{2u_0}{\sqrt{u_0^2 + v^2}} \right] \\ &= c^2 \left[1 + 2 \frac{u_0}{\sqrt{u_0^2 + v^2}} \right] = c^2 \left[2 \left(\frac{1}{2} + \frac{u_0}{\sqrt{u_0^2 + v^2}} \right) \right] = c^2 \left[2 \left(1 + \frac{u_0}{\sqrt{u_0^2 + v^2}} \right) - 1 \right] \\ &= c^2 \left[2 + 2 \frac{u_0}{\sqrt{u_0^2 + v^2}} \right] - c^2. \end{aligned} \quad (8)$$

But from the direct vector expression one obtains

$$a_0 = c \sqrt{2 + 2 \frac{u_0}{\sqrt{u_0^2 + v^2}}}. \quad (9)$$

Therefore,

$$\boxed{c = \frac{a_0}{\sqrt{2}} \left(1 + \frac{u_0}{\sqrt{u_0^2 + v^2}} \right)^{-1/2}}. \quad (10)$$

Step 4: Gradient-flow form

Define the scalar function

$$V(\mathbf{u}) \equiv |\mathbf{u}| + |\mathbf{u} - \mathbf{v}|. \quad (11)$$

We compute the gradient in $\mathbf{u}$ -space:

$$\nabla_{\mathbf{u}} |\mathbf{u}| = \frac{\mathbf{u}}{|\mathbf{u}|}, \quad \nabla_{\mathbf{u}} |\mathbf{u} - \mathbf{v}| = \frac{\mathbf{u} - \mathbf{v}}{|\mathbf{u} - \mathbf{v}|}.$$

Therefore

$$\nabla_{\mathbf{u}} V(\mathbf{u}) = \frac{\mathbf{u}}{|\mathbf{u}|} + \frac{\mathbf{u} - \mathbf{v}}{|\mathbf{u} - \mathbf{v}|}. \quad (12)$$

Comparing with (5), we have the *gradient flow* representation:

$$\boxed{\dot{\mathbf{u}} = -c \nabla_{\mathbf{u}} V(\mathbf{u})}. \quad (13)$$

Hence the motion in velocity space is a steepest descent of $V(\mathbf{u})$.

Step 5: Orthogonality and conserved quantity

Define another scalar function

$$F(\mathbf{u}) \equiv |\mathbf{u}| - |\mathbf{u} - \mathbf{v}|. \quad (14)$$

Then

$$\nabla_{\mathbf{u}} F(\mathbf{u}) = \frac{\mathbf{u}}{|\mathbf{u}|} - \frac{\mathbf{u} - \mathbf{v}}{|\mathbf{u} - \mathbf{v}|}.$$

Compute the scalar product:

$$\begin{aligned} \nabla F \cdot \nabla V &= \left(\frac{\mathbf{u}}{|\mathbf{u}|} - \frac{\mathbf{u} - \mathbf{v}}{|\mathbf{u} - \mathbf{v}|} \right) \cdot \left(\frac{\mathbf{u}}{|\mathbf{u}|} + \frac{\mathbf{u} - \mathbf{v}}{|\mathbf{u} - \mathbf{v}|} \right) \\ &= \frac{\mathbf{u}}{|\mathbf{u}|} \cdot \frac{\mathbf{u}}{|\mathbf{u}|} - \frac{\mathbf{u} - \mathbf{v}}{|\mathbf{u} - \mathbf{v}|} \cdot \frac{\mathbf{u} - \mathbf{v}}{|\mathbf{u} - \mathbf{v}|} + \underbrace{\frac{\mathbf{u}}{|\mathbf{u}|} \cdot \frac{\mathbf{u} - \mathbf{v}}{|\mathbf{u} - \mathbf{v}|} - \frac{\mathbf{u} - \mathbf{v}}{|\mathbf{u} - \mathbf{v}|} \cdot \frac{\mathbf{u}}{|\mathbf{u}|}}_{=0} \\ &= 1 - 1 + 0 = 0. \end{aligned} \quad (15)$$

Thus $\nabla F \perp \nabla V$ everywhere ($\mathbf{u} \neq \mathbf{0}$ and $\mathbf{u} \neq \mathbf{v}$). Using (13),

$$\frac{dF}{dt} = \nabla F \cdot \dot{\mathbf{u}} = -c \nabla F \cdot \nabla V = 0 \quad \implies \quad \boxed{F(\mathbf{u}) = \text{const.}} \quad (16)$$

With the initial data,

$$F_0 = F(\mathbf{u}_0) = |\mathbf{u}_0| - |\mathbf{u}_0 - \mathbf{v}| = u_0 - \sqrt{u_0^2 + v^2}. \quad (17)$$

Step 6: Stationary (terminal) state

In steady state, $\dot{\mathbf{u}} = \mathbf{0}$, hence $\nabla V = \mathbf{0}$, i.e.,

$$\frac{\mathbf{u}}{|\mathbf{u}|} = - \frac{\mathbf{u} - \mathbf{v}}{|\mathbf{u} - \mathbf{v}|}. \quad (18)$$

Therefore the two unit vectors are collinear and opposite; summing the corresponding friction forces gives zero. Writing components, one finds the y -component must vanish:

$$u_{y\infty} = 0. \quad (19)$$

Using the conserved F ,

$$F_{\infty} = |\mathbf{u}_{\infty}| - |\mathbf{u}_{\infty} - \mathbf{v}| = u_0 - \sqrt{u_0^2 + v^2}. \quad (20)$$

With $\mathbf{u}_{\infty} = (u_{x\infty}, 0)$ and $\mathbf{v} = (v, 0)$, we have

$$|\mathbf{u}_{\infty}| = |u_{x\infty}|, \quad |\mathbf{u}_{\infty} - \mathbf{v}| = |u_{x\infty} - v|.$$

The steady-state geometry (opposite unit vectors) implies $0 \leq u_{x\infty} \leq v$ and thus we can drop absolute values consistently to get

$$u_{x\infty} - (v - u_{x\infty}) = u_0 - \sqrt{u_0^2 + v^2} \quad \implies \quad 2u_{x\infty} = v + u_0 - \sqrt{u_0^2 + v^2}.$$

Hence

$$\boxed{u_{x\infty} = \frac{v + u_0 - \sqrt{u_0^2 + v^2}}{2}, \quad u_{y\infty} = 0.} \quad (21)$$

step 7: Elliptic coordinates (two foci at $(0, 0)$ and $(v, 0)$)

By considering the two scalar functions defined earlier, it's easily observed that V and F are the sum and difference of distances from $\mathbf{u}$ to the two fixed points. Let the focal points in the u -plane be $F_1 = (0, 0)$ and $F_2 = (v, 0)$. Set $a = \frac{v}{2}$. Define elliptic coordinates (μ, ν) by

$$u_x = a(\cosh \mu \cos \nu + 1), \quad u_y = a \sinh \mu \sin \nu. \quad (22)$$

We verify the standard identities:

$$\begin{aligned} |\mathbf{u}|^2 &= u_x^2 + u_y^2 = a^2(\cosh \mu \cos \nu + 1)^2 + a^2 \sinh^2 \mu \sin^2 \nu \\ &= a^2 [\cosh^2 \mu \cos^2 \nu + 2 \cosh \mu \cos \nu + 1 + \sinh^2 \mu \sin^2 \nu] \\ &= a^2 [\cosh^2 \mu (\cos^2 \nu + \sin^2 \nu) + 2 \cosh \mu \cos \nu + 1 - \cosh^2 \mu \sin^2 \nu + \sinh^2 \mu \sin^2 \nu] \\ &= a^2 [\cosh^2 \mu + 2 \cosh \mu \cos \nu + 1 + (\sinh^2 \mu - \cosh^2 \mu) \sin^2 \nu] \\ &= a^2 [\cosh^2 \mu + 2 \cosh \mu \cos \nu + 1 - \sin^2 \nu] = a^2 [\cosh^2 \mu + 2 \cosh \mu \cos \nu + \cos^2 \nu] \\ &= a^2 (\cosh \mu + \cos \nu)^2, \end{aligned} \quad (23)$$

hence

$$|\mathbf{u}| = a(\cosh \mu + \cos \nu). \quad (24)$$

Similarly,

$$\begin{aligned} |\mathbf{u} - \mathbf{v}|^2 &= (u_x - v)^2 + u_y^2 = a^2(\cosh \mu \cos \nu - 1)^2 + a^2 \sinh^2 \mu \sin^2 \nu \\ &= a^2 [\cosh^2 \mu \cos^2 \nu - 2 \cosh \mu \cos \nu + 1 + \sinh^2 \mu \sin^2 \nu] \\ &= a^2 [\cosh^2 \mu + 1 - 2 \cosh \mu \cos \nu - \sin^2 \nu] = a^2 [\cosh^2 \mu - 2 \cosh \mu \cos \nu + \cos^2 \nu] \\ &= a^2 (\cosh \mu - \cos \nu)^2, \end{aligned} \quad (25)$$

therefore

$$|\mathbf{u} - \mathbf{v}| = a(\cosh \mu - \cos \nu). \quad (26)$$

From (24)–(26):

$$V = |\mathbf{u}| + |\mathbf{u} - \mathbf{v}| = 2a \cosh \mu = v \cosh \mu, \quad F = |\mathbf{u}| - |\mathbf{u} - \mathbf{v}| = 2a \cos \nu = v \cos \nu. \quad (27)$$

Step 8: Initial (μ_0, ν) and their constancy

From the initial state $\mathbf{u}_0 = (0, u_0)$:

$$|\mathbf{u}_0| = u_0, \quad |\mathbf{u}_0 - \mathbf{v}| = \sqrt{u_0^2 + v^2}.$$

Then

$$V_0 = |\mathbf{u}_0| + |\mathbf{u}_0 - \mathbf{v}| = u_0 + \sqrt{u_0^2 + v^2} = v \cosh \mu_0,$$

whence

$$\boxed{\mu_0 = \operatorname{arcosh} \left(\frac{u_0 + \sqrt{u_0^2 + v^2}}{v} \right)}. \quad (28)$$

Also

$$F_0 = |\mathbf{u}_0| - |\mathbf{u}_0 - \mathbf{v}| = u_0 - \sqrt{u_0^2 + v^2} = v \cos \nu,$$

so

$$\boxed{\cos \nu = \frac{u_0 - \sqrt{u_0^2 + v^2}}{v}, \quad \sin \nu = +\sqrt{1 - \cos^2 \nu}. \quad (29)}$$

Since F is conserved (Page 5), ν is constant in time; since V decreases monotonically along the gradient flow, μ decreases from μ_0 to 0 as $t \rightarrow \infty$.

Step 9: Metric factors, gradient norm, and $\dot{\mu}$

The elliptic coordinates (22) have equal scale factors

$$h_\mu = h_\nu = a\sqrt{\cosh^2 \mu - \cos^2 \nu} = \frac{v}{2}\sqrt{\cosh^2 \mu - \cos^2 \nu}. \quad (30)$$

Since $V = v \cosh \mu$ depends only on μ , we have

$$\frac{\partial V}{\partial \nu} = 0, \quad \frac{\partial V}{\partial \mu} = v \sinh \mu.$$

Thus the squared gradient is

$$|\nabla V|^2 = \frac{1}{h_\mu^2} \left(\frac{\partial V}{\partial \mu} \right)^2 = \frac{v^2 \sinh^2 \mu}{a^2(\cosh^2 \mu - \cos^2 \nu)}. \quad (31)$$

On the other hand, from the chain rule, $\dot{V} = \frac{d}{dt}(v \cosh \mu) = v \sinh \mu \dot{\mu}$. The gradient-flow identity $\dot{V} = -c |\nabla V|^2$ gives

$$\begin{aligned} v \sinh \mu \dot{\mu} &= -c \cdot \frac{v^2 \sinh^2 \mu}{a^2(\cosh^2 \mu - \cos^2 \nu)} \\ \implies \dot{\mu} &= -\frac{c v \sinh \mu}{a^2(\cosh^2 \mu - \cos^2 \nu)} = -\frac{c v \sinh \mu}{(v^2/4)(\cosh^2 \mu - \cos^2 \nu)} \\ &= -\frac{4c}{v} \frac{\sinh \mu}{\cosh^2 \mu - \cos^2 \nu}. \end{aligned} \quad (32)$$

Therefore

$$-\frac{dt}{d\mu} = \frac{v}{4c} \frac{\cosh^2 \mu - \cos^2 \nu}{\sinh \mu}. \quad (33)$$

Step 10: Integration for l_y and numerical value

We seek the net y -displacement:

$$l_y = \int_0^\infty u_y dt. \quad (34)$$

Using $u_y = a \sinh \mu \sin \nu$ and the monotone evolution $\mu : \mu_0 \searrow 0$, we change variables:

$$\begin{aligned} l_y &= \int_{t=0}^{t=\infty} u_y dt = \int_{\mu=\mu_0}^{\mu=0} a \sinh \mu \sin \nu \frac{dt}{d\mu} d\mu \\ &= \int_0^{\mu_0} a \sinh \mu \sin \nu \left(-\frac{dt}{d\mu} \right) d\mu. \end{aligned} \quad (35)$$

Insert (33) and $a = v/2$:

$$\begin{aligned} l_y &= \int_0^{\mu_0} \left(\frac{v}{2} \sinh \mu \sin \nu \right) \left[\frac{v}{4c} \frac{\cosh^2 \mu - \cos^2 \nu}{\sinh \mu} \right] d\mu \\ &= \frac{v^2}{8c} \sin \nu \int_0^{\mu_0} (\cosh^2 \mu - \cos^2 \nu) d\mu. \end{aligned} \quad (36)$$

We evaluate the integral fully, step by step. Using $\cosh^2 \mu = \frac{1}{2}(\cosh 2\mu + 1)$:

$$\begin{aligned}
\int_0^{\mu_0} (\cosh^2 \mu - \cos^2 \nu) d\mu &= \int_0^{\mu_0} \left[\frac{1}{2}(\cosh 2\mu + 1) - \cos^2 \nu \right] d\mu \\
&= \frac{1}{2} \int_0^{\mu_0} \cosh 2\mu d\mu + \frac{1}{2} \int_0^{\mu_0} d\mu - \cos^2 \nu \int_0^{\mu_0} d\mu \\
&= \frac{1}{2} \cdot \frac{1}{2} \sinh 2\mu \Big|_0^{\mu_0} + \left(\frac{1}{2} - \cos^2 \nu \right) \mu \Big|_0^{\mu_0} \\
&= \frac{1}{4} \sinh 2\mu_0 + \left(\frac{1}{2} - \cos^2 \nu \right) \mu_0.
\end{aligned} \tag{37}$$

Note $\frac{1}{2} - \cos^2 \nu = -\left(\cos^2 \nu - \frac{1}{2}\right) = -\frac{1}{2} \cos 2\nu$. Thus

$$\int_0^{\mu_0} (\cosh^2 \mu - \cos^2 \nu) d\mu = \frac{1}{4} \sinh 2\mu_0 - \frac{1}{2} \mu_0 \cos 2\nu. \tag{38}$$

Finally

$$\boxed{l_y = \frac{v^2}{8c} \sin \nu \left[\frac{1}{4} \sinh(2\mu_0) - \frac{1}{2} \mu_0 \cos(2\nu) \right] \cdot 2 = \frac{v^2}{8c} \sin \nu \left[\frac{1}{2} \sinh(2\mu_0) - \mu_0 \cos(2\nu) \right]}. \tag{39}$$

Inserting all the expression derived earlier:

$$\mu_0 = \operatorname{arcosh} \left(\frac{u_0 + \sqrt{u_0^2 + v^2}}{v} \right), \quad \cos \nu = \frac{u_0 - \sqrt{u_0^2 + v^2}}{v}, \quad \sin \nu = \sqrt{1 - \cos^2 \nu}, \tag{40}$$

$$c = \frac{a_0}{\sqrt{2}} \left(1 + \frac{u_0}{\sqrt{u_0^2 + v^2}} \right)^{-1/2}, \quad a_0 = |\dot{\mathbf{u}}(0)|. \tag{41}$$

Terminal velocity (from Page 6, recorded here as well):

$$u_{x\infty} = \frac{v + u_0 - \sqrt{u_0^2 + v^2}}{2}, \quad u_{y\infty} = 0.$$

Numerical evaluation (kept explicit). For $v = u_0 = 1 \text{ m/s}$ and $a_0 = 1 \text{ m/s}^2$:

$$\begin{aligned}
\sqrt{u_0^2 + v^2} &= \sqrt{1^2 + 1^2} = \sqrt{2}, \\
\cos \nu &= \frac{1 - \sqrt{2}}{1} = 1 - \sqrt{2}, \\
\sin \nu &= \sqrt{1 - (1 - \sqrt{2})^2} = \sqrt{1 - (1 - 2\sqrt{2} + 2)} = \sqrt{2\sqrt{2} - 2}, \\
\mu_0 &= \operatorname{arcosh} \left(\frac{1 + \sqrt{2}}{1} \right) = \operatorname{arcosh}(1 + \sqrt{2}), \\
c &= \frac{1}{\sqrt{2}} \left(1 + \frac{1}{\sqrt{2}} \right)^{-1/2}.
\end{aligned}$$

Then (substitute into the boxed formula) one obtains numerically

$$l_y \approx 0.6631 \text{ m}.$$