

PhysicsCup 2026 – Problem 5

Crystal Ball

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Solution.

Let us call the radius of the ball R , its refractive index n and the distance from the front point of the sphere to the screen L . We also set

$$\lambda = \frac{L}{R}.$$

Let us put the source at the rear point of the sphere,

$$S = (-R, 0),$$

and let a ray leave the source under angle α with the axis. Then the ray is

$$(x, y) = (-R, 0) + t(\cos \alpha, \sin \alpha).$$

Its second intersection with the sphere $x^2 + y^2 = R^2$ is at

$$t = 2R \cos \alpha,$$

so the hit point is

$$P = R(\cos 2\alpha, \sin 2\alpha).$$

So the incidence angle at the surface must be exactly α .

Let us call the refracted angle to the normal β . By Snell's law,

$$\beta(\alpha) = \sin^{-1}(n \sin \alpha).$$

So the outgoing ray makes angle

$$2\alpha - \beta(\alpha)$$

with the axis. Therefore its hit position on the screen is

$$u(\alpha) \equiv \frac{y_{\text{scr}}}{R} = \sin 2\alpha + (\lambda + 1 - \cos 2\alpha) \tan(2\alpha - \beta(\alpha)).$$

Now let us use the picture. With the white segment $d = 2R$ as scale, we measure

$$u_1 = \frac{D_{\text{in}}}{d} \approx 0.291, \quad u_2 = \frac{D_{\text{out}}}{d} \approx 1.384.$$

The small bright circle is a caustic, so for some α_* we have

$$u(\alpha_*) = u_1, \quad u'(\alpha_*) = 0.$$

The outer edge comes from the critical ray. So

$$\alpha_c = \sin^{-1} \frac{1}{n}, \quad |u(\alpha_c)| = u_2.$$

Let us solve these three equations. We get

$$\alpha_* \approx 0.270, \quad n \approx 2.17, \quad \lambda \approx 2.69.$$

Now let us find the wanted fraction. The impact parameter is

$$b = R \sin \alpha,$$

and this stays the same at every reflection. So the incidence angle at every later hit is again α . Therefore:

$$\alpha < \alpha_c \Rightarrow \text{the ray can leave directly,}$$

$$\alpha > \alpha_c \Rightarrow \text{the ray is trapped by total internal reflection until scattering.}$$

Since there is no absorption, every trapped ray finally leaves as scattered light.

Let us compute the direct part: The directly escaping rays are exactly the rays inside two cones of half-angle α_c . So

$$f_{\text{dir}} = \frac{2 \cdot 2\pi(1 - \cos \alpha_c)}{4\pi} = 1 - \cos \alpha_c.$$

So

$$f_{\text{sc}} = 1 - f_{\text{dir}} = \cos \alpha_c = \sqrt{1 - \frac{1}{n^2}}.$$

With $n \approx 2.17$, this gives us

$$f_{\text{sc}} = \sqrt{1 - \frac{1}{2.17^2}} \approx 0.888.$$

$$\boxed{f_{\text{sc}} \approx 0.89}$$

So about 89% of the emitted light leaves the ball as scattered light.