

Physics Cup 2025 - Solution to Problem 4

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I)

Let's start with the general assumptions. We will work in the co-rotating frame. Gravity is neglected. The container height is $h = d$, $R \gg d$. Surface tension is assumed large, $\sigma \gg \rho d^2 R \Omega^2$, and the contact angle is 90° . Under these conditions the meniscus in each square hole remains intact and nearly planar, and its Laplace pressure jump is approximately a constant for small translations. Therefore, to linear order in the interface displacement, the pressure on the air side may be taken as p_{atm} and capillarity contributes no linear term proportional to x [7].

Water is treated as incompressible and the chamber/walls are rigid on the time scale of interest, so the total water volume is constant. While $x \ll d$, the interfaces do not break and no air enters the main chamber, hence the net volume change across the holes must vanish:

$$\sum_i h^2 x_i = \text{const} \implies \sum_i \dot{x}_i = 0, \quad (1)$$

and in the partitioned configuration (1) holds separately in each semi-compartment.

But how is $\sum_i \dot{x}_i = 0$ possible if the system is symmetric? The key point is that the geometric symmetry only implies that the **equations** are invariant under permutations/rotations of the holes; it does **not** force the evolving solution to remain uniform. In fact, the perfectly symmetric motion $x_1 = \dots = x_6$ would violate (1) unless $\dot{x}_i = 0$, so any nontrivial motion must satisfy $\sum_i x_i = 0$ and therefore necessarily involves both signs of $\dot{x}_i$ (some holes have outward interface motion, others inward). The non-uniform pattern that forms is selected by the initial perturbation (imperfections/noise) and, in the unstable regime, the fastest-growing admissible eigenmode quickly dominates the dynamics.

II)

Let $x_i(t)$ be the small outward displacement of the water-air interface in hole i , measured along the radial hole axis. Let $p_i(t)$ be the pressure in the main chamber at the inner mouth ($r = R$) of hole i .

In the rotating frame, the radial momentum equation (neglecting convective terms at small amplitude) is [1, 2]

$$\rho \partial_t u = -\partial_r p + \rho \Omega^2 r,$$

where the velocity in the short hole is approximated as uniform, $u(t) \approx \dot{x}_i(t)$. Integrating from $r = R$ to the interface at $r = R + d + x_i$ gives

$$p_{\text{atm}} - p_i = \rho \Omega^2 \int_R^{R+d+x_i} r \, dr - \rho \int_R^{R+d+x_i} \ddot{x}_i \, dr.$$

Immediately after unplugging there exists a static reference state in the rotating frame with $x_i = 0$ for all holes and $\ddot{x}_i = 0$, in which the meniscus boundary condition fixes the additive pressure constant so that $p(R + d) = p_{\text{atm}}$. Writing $p_i = p_i^{(0)} + \delta p_i$ and linearizing about this equilibrium (keeping only $O(x_i)$ terms and using $R \gg d$) yields

$$\rho \ell \ddot{x}_i = \delta p_i + \rho \Omega^2 R x_i, \quad (2)$$

where ℓ is the effective moving length of the water column in the hole (for the fill $r < R + d$ one has $\ell \simeq d$; keeping ℓ symbolic will make the cancellation in $T_2^2 - T_1^2$ explicit). The term $+\rho \Omega^2 R x_i$ has the form of an inverted spring, producing exponential growth away from $x_i = 0$ when δp_i is determined self-consistently.

III)

Because the chamber height is $h = d \ll R$, the main chamber motion is effectively two-dimensional with depth h (very thin disk), so we disregard motions along the rotation axis. Also, from (2), we can see that exponential rates $x \sim e^{\lambda t}$ will appear of the order of

$$\lambda^2 \sim \frac{\Omega^2 R}{\ell} \quad \Rightarrow \quad \lambda \sim \Omega \sqrt{\frac{R}{d}} \gg \Omega, \quad (3)$$

which means that we can also neglect the Coriolis term at leading order (since $\|2\Omega \times \mathbf{u}\|/\|\partial_t \mathbf{u}\| \sim \Omega/\lambda \sim \sqrt{h/R} \ll 1$). Thus, the chamber flow is treated as irrotational potential flow [1, 3]:

$$\mathbf{u} = \nabla \phi, \quad \nabla^2 \phi = 0 \quad (r < R).$$

A hole of cross-section $h \times h$ carries volume flux $Q_i = h^2 \dot{x}_i$, so per unit depth the corresponding 2D flux is

$$q_i \equiv \frac{Q_i}{h} = h \dot{x}_i.$$

Since the boundary normal velocity is nonzero only over the small hole arcs (width $\sim h$), and $h \ll R$, each hole is approximated as a localized *Neumann* source at the boundary angle θ_i : prescribing the normal velocity on $r = R$ is equivalent to prescribing the Neumann data

$$u_n(R, \theta) \equiv \mathbf{u} \cdot \hat{\mathbf{n}} = \left. \frac{\partial \phi}{\partial n} \right|_{r=R} = \left. \frac{\partial \phi}{\partial r} \right|_{r=R} =: g(\theta), \quad (4)$$

and the small-arc flux injection/extraction carried by hole i is represented (in the $h \ll R$ limit) by a delta-function contribution of total strength q_i ,

$$g(\theta) \simeq \sum_i \frac{q_i}{R} \delta(\theta - \theta_i), \quad (5)$$

up to an (irrelevant) constant component. The “zero-mean” condition for the Neumann problem is the solvability constraint

$$\int_0^{2\pi} g(\theta) R d\theta = 0 \quad \Longleftrightarrow \quad \sum_i q_i = 0, \quad (6)$$

which holds here precisely because of the volume constraint $\sum_i q_i = h \sum_i \dot{x}_i = 0$ from (1).

Electrostatics analogy: The potential ϕ plays the role of an electric potential V in a charge-free region, so both satisfy Laplace’s equation. Prescribing $\partial_n \phi$ on the boundary (the Neumann condition) is analogous to prescribing the normal electric field $E_n = -\partial_n V$, i.e. fixing the flux through the boundary rather than the potential value. A localized Neumann source is then

analogous to a localized injection of flux (a pointlike source in the boundary flux distribution). The requirement of zero-mean Neumann data corresponds to the fact that the total injected flux must vanish (no net “charge”/flux creation in the closed domain).

To compute the boundary potential induced by (5), we solve Laplace’s equation in the disk by Fourier modes. A harmonic function regular at $r = 0$ admits the expansion

$$\phi(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta), \quad (7)$$

so the radial derivative at $r = R$ is

$$\left. \frac{\partial \phi}{\partial r} \right|_{r=R} = \sum_{n=1}^{\infty} n R^{n-1} (A_n \cos n\theta + B_n \sin n\theta) = g(\theta). \quad (8)$$

For a localized source at θ_i one uses the Fourier representation

$$\delta(\theta - \theta_i) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \cos(n(\theta - \theta_i)), \quad (9)$$

and the constant term $1/(2\pi)$ is precisely the mean component removed by (6). Matching Fourier coefficients in (8) then implies that, on the boundary $r = R$, the contribution of hole i to the potential has the form

$$\phi_i(R, \theta) = -\frac{q_i}{\pi} \sum_{n=1}^{\infty} \frac{\cos(n(\theta - \theta_i))}{n} + (\text{constant}), \quad (10)$$

so summing over holes gives $\phi = \sum_i \phi_i$. The Fourier series in (10) is evaluated with the standard identity [5, 6]

$$\sum_{n=1}^{\infty} \frac{\cos(n\Delta)}{n} = -\ln(2|\sin(\Delta/2)|), \quad (11)$$

which yields the logarithmic boundary kernel (Neumann Green function on the disk) [3, 4]:

$$\phi(R, \theta_j) = -\frac{1}{\pi} \sum_i q_i \ln\left(2\left|\sin \frac{\theta_j - \theta_i}{2}\right|\right) + (\text{additive const.}). \quad (12)$$

The logarithm diverges as $\theta_j \rightarrow \theta_i$, reflecting the idealization of a finite-width hole by a point source. In reality each hole occupies an arc of width $\Delta\theta \sim h/R$, and averaging (12) over that arc replaces the divergent “self” contribution by a finite constant L_0 depending on the hole geometry. This replacement is called **self-regularization**.

Finally, linear unsteady Bernoulli for potential flow gives [1, 2]

$$\delta p = -\rho \partial_t \phi. \quad (13)$$

Evaluating (13) at $\theta = \theta_j$ and using $q_i = h\dot{x}_i$ (so $\dot{q}_i = h\ddot{x}_i$) gives

$$\delta p_j = \rho \frac{h}{\pi} \sum_i L_{ji} \ddot{x}_i, \quad L_{ji} = \begin{cases} \ln\left(2\left|\sin \frac{\theta_j - \theta_i}{2}\right|\right), & i \neq j, \\ L_0, & i = j, \end{cases} \quad (14)$$

where the diagonal entry L_{jj} is precisely the self-regularized constant introduced above. Combining (2) and (14) yields

$$\sum_i \left[\ell \delta_{ji} - \frac{h}{\pi} L_{ji} \right] \ddot{x}_i = \Omega^2 R x_j, \quad (15)$$

and it is convenient to define the effective mass matrix

$$\mathcal{M}_{ji} \equiv \ell \delta_{ji} - \frac{h}{\pi} L_{ji}.$$

IV)

Equation (15) is a linear system with constant coefficients. Restricting to the admissible subspace enforced by (1), we may expand the solution in eigenmodes

$$x(t) = \sum_{\alpha} c_{\alpha} v^{(\alpha)} e^{\lambda_{\alpha} t}, \quad \lambda_{\alpha}^2 = \frac{\Omega^2 R}{m_{\alpha}}, \quad (16)$$

where m_{α} and $v^{(\alpha)}$ are eigenpairs of $\mathcal{M}$. If the system is unstable, several λ_{α} are positive, and for generic initial data the largest λ_{α} dominates after a transient [8]:

$$x(t) \sim c_{\max} v^{(\max)} e^{\lambda_{\max} t}. \quad (17)$$

The experiment defines the measurement origin in the regime where the motion is already well approximated by this exponential, so the doubling time from x to $2x$ is

$$T = \frac{\ln 2}{\lambda_{\max}} \implies T^2 = (\ln 2)^2 \frac{m_{\min}}{\Omega^2 R}, \quad (18)$$

where $m_{\min}$ is the smallest admissible eigenvalue of $\mathcal{M}$ (equivalently the largest admissible eigenvalue of L).

V)

In the unpartitioned case the six holes sit at equally spaced angles

$$\theta_k = \frac{k\pi}{3}, \quad k = 0, 1, 2, 3, 4, 5.$$

Recall from (14) that, for $i \neq j$, the coupling matrix entries are determined by the boundary kernel

$$L_{ji} = L(\theta_j - \theta_i), \quad L(\Delta) \equiv \ln(2|\sin(\Delta/2)|), \quad (\Delta \neq 0), \quad (19)$$

while the diagonal is the self-regularized constant $L_{jj} = L_0$. Since the holes are equally spaced, the only nonzero separations (mod 2π) between distinct holes are

$$\Delta\theta \in \left\{ \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}, \pi \right\},$$

so it is convenient to abbreviate

$$L\left(\frac{\pi}{3}\right), \quad L\left(\frac{2\pi}{3}\right), \quad L(\pi),$$

which are computed directly from (19):

$$L\left(\frac{\pi}{3}\right) = \ln\left(2 \sin \frac{\pi}{6}\right) = \ln(1) = 0, \quad L\left(\frac{2\pi}{3}\right) = \ln\left(2 \sin \frac{\pi}{3}\right) = \ln(\sqrt{3}),$$

$$L(\pi) = \ln\left(2 \sin \frac{\pi}{2}\right) = \ln 2.$$

Because $L(\Delta)$ depends only on the angular difference $\theta_j - \theta_i$ and the set of holes is invariant under a cyclic shift $k \mapsto k + 1$, the matrix L has the form

$$L_{jk} = c_{(j-k) \bmod 6}, \quad (20)$$

i.e. each row is obtained from the previous one by a cyclic shift. A matrix with the property (20) is called **circulant**. For an $N \times N$ circulant matrix with first row $(c_0, c_1, \dots, c_{N-1})$, the discrete Fourier vectors

$$v_k^{(m)} = e^{i2\pi mk/N}, \quad m = 0, 1, \dots, N-1, \quad (21)$$

are eigenvectors, with eigenvalues given by the discrete Fourier transform of the first row,

$$\lambda_L(m) = \sum_{p=0}^{N-1} c_p e^{-i2\pi mp/N}. \quad (22)$$

In the present case $N = 6$ and, using the symmetry $c_p = c_{6-p}$, it is convenient to write the real form of (22):

$$\lambda_L(m) = c_0 + 2c_1 \cos\left(\frac{2\pi m}{6}\right) + 2c_2 \cos\left(\frac{4\pi m}{6}\right) + c_3 \cos(\pi m), \quad (23)$$

where the coefficients c_p are read off from the separations:

$$c_0 = L_0, \quad c_1 = L\left(\frac{\pi}{3}\right) = 0, \quad c_2 = L\left(\frac{2\pi}{3}\right) = \ln(\sqrt{3}), \quad c_3 = L(\pi) = \ln 2.$$

Substituting these values into (23) yields

$$\lambda_L(m) = L_0 + 2L\left(\frac{\pi}{3}\right) \cos\left(\frac{2\pi m}{6}\right) + 2L\left(\frac{2\pi}{3}\right) \cos\left(\frac{4\pi m}{6}\right) + L(\pi) \cos(\pi m), \quad (24)$$

and using $L(\pi/3) = 0$ gives the simplified expression

$$\lambda_L(m) = L_0 + 2 \ln(\sqrt{3}) \cos\left(\frac{2\pi m}{3}\right) + \ln 2 (-1)^m. \quad (25)$$

The volume constraint (1) restricts admissible mode shapes to those with

$$\sum_{k=0}^5 v_k = 0.$$

The Fourier mode $m = 0$ has $v_k^{(0)} \equiv 1$, hence $\sum_k v_k^{(0)} \neq 0$, so it is excluded. For $m \neq 0$ one has $\sum_{k=0}^5 e^{i2\pi mk/6} = 0$, so the remaining Fourier modes are admissible.

To identify the dominant (fastest-growing) admissible mode, note from (15) that m enters through the effective mass eigenvalue

$$m^{(\alpha)} = \ell - \frac{h}{\pi} \lambda_L^{(\alpha)}.$$

Thus, among admissible modes, the **largest** λ_L produces the **smallest** effective mass and therefore the **largest** growth rate (since $\lambda^2 = \Omega^2 R/m$). Using (25) for $m = 1, 2, 3, 4, 5$ and the symmetry $\lambda_L(m) = \lambda_L(6-m)$, we find

$$\lambda_L(1) = \lambda_L(5) = L_0 - \ln(\sqrt{3}) - \ln 2, \quad \lambda_L(2) = \lambda_L(4) = L_0 - \ln(\sqrt{3}) + \ln 2,$$

$$\lambda_L(3) = L_0 + 2 \ln(\sqrt{3}) - \ln 2,$$

so the maximal admissible eigenvalue occurs at $m = 3$. The corresponding eigenvector is

$$v_k^{(3)} = e^{i\pi k} = (-1)^k, \quad (26)$$

i.e. the alternating pattern. Therefore the relevant effective mass eigenvalue is

$$m_1 = \ell - \frac{h}{\pi} \left(L_0 + 2 \ln(\sqrt{3}) - \ln 2 \right). \quad (27)$$

VI)

In the partitioned case, a radial partition along the 12-6 diameter splits the chamber into two semi-disks. Consider the right semi-compartment, which contains the three holes at

$$\theta_1 = -\frac{\pi}{3}, \quad \theta_2 = 0, \quad \theta_3 = \frac{\pi}{3}.$$

The partition is a rigid wall, hence it is a no-flux boundary:

$$\mathbf{u} \cdot \hat{\mathbf{n}} = 0 \quad \Longleftrightarrow \quad \frac{\partial \phi}{\partial n} = 0 \quad \text{on the diameter.} \quad (28)$$

Therefore the potential ϕ in the half-disk satisfies Laplace's equation with a Neumann condition on both the circular boundary $r = R$ (where the holes inject/extract flux) and the straight boundary (the partition).

To enforce the Neumann condition on the partition, we use the method of images [4, 3]. For a Neumann boundary at a line, the image source has the **same** sign. Concretely, reflect each boundary source at angle θ_i across the partition (the vertical diameter) to an image source located at

$$\theta'_i = \pi - \theta_i, \quad (29)$$

with the same flux strength q_i . Then the normal derivative of the resulting potential vanishes on the partition by symmetry, because the normal components contributed by a source and its mirror image cancel at the boundary (where the line is the perpendicular bisector of the pair). In this way, the half-disk Neumann problem is reduced to a full-disk problem with 3 physical sources plus 3 same-sign images.

Since the full-disk boundary kernel is $L(\Delta) = \ln(2|\sin(\Delta/2)|)$ (as in (14)), the effective coupling between physical holes in the half-disk is obtained by adding the contribution from each physical source and its image:

$$L_{ji}^{(\text{half})} \equiv L^{(\text{half})}(\theta_j, \theta_i) = L(\theta_j - \theta_i) + L(\theta_j - \theta'_i), \quad \theta'_i = \pi - \theta_i, \quad (30)$$

where, as before, the diagonal entries are self-regularized:

$$L_{jj}^{(\text{half})} = L_0 \quad (\text{finite, set by the hole width}).$$

We now compute the entries explicitly. Define

$$a \equiv \ln(\sqrt{3}), \quad b \equiv \ln 2.$$

First note the required kernel values from (19):

$$L\left(\frac{\pi}{3}\right) = 0, \quad L\left(\frac{2\pi}{3}\right) = a, \quad L(\pi) = b.$$

Using (30) and the angles $(\theta_1, \theta_2, \theta_3) = (-\pi/3, 0, \pi/3)$:

VI-I)

Off-diagonal entry $j = 1, i = 2$:

$$\theta'_2 = \pi - \theta_2 = \pi, \quad L_{12}^{(\text{half})} = L(\theta_1 - \theta_2) + L(\theta_1 - \theta'_2) = L\left(-\frac{\pi}{3}\right) + L\left(-\frac{4\pi}{3}\right) = 0 + a = a.$$

VI-II)

Off-diagonal entry $j = 2, i = 1$:

$$\theta'_1 = \pi - \theta_1 = \frac{4\pi}{3}, \quad L_{21}^{(\text{half})} = L(\theta_2 - \theta_1) + L(\theta_2 - \theta'_1) = L\left(\frac{\pi}{3}\right) + L\left(-\frac{4\pi}{3}\right) = 0 + a = a.$$

VI-III)

Off-diagonal entry $j = 2, i = 3$ (by symmetry it will match $L_{21}^{(\text{half})}$):

$$\theta'_3 = \pi - \theta_3 = \frac{2\pi}{3}, \quad L_{23}^{(\text{half})} = L(\theta_2 - \theta_3) + L(\theta_2 - \theta'_3) = L\left(-\frac{\pi}{3}\right) + L\left(-\frac{2\pi}{3}\right) = 0 + a = a.$$

VI-IV)

Off-diagonal entry $j = 1, i = 3$:

$$\theta'_3 = \frac{2\pi}{3}, \quad L_{13}^{(\text{half})} = L(\theta_1 - \theta_3) + L(\theta_1 - \theta'_3) = L\left(-\frac{2\pi}{3}\right) + L(-\pi) = a + b.$$

VI-V)

Off-diagonal entry $j = 3, i = 1$:

$$\theta'_1 = \frac{4\pi}{3}, \quad L_{31}^{(\text{half})} = L(\theta_3 - \theta_1) + L(\theta_3 - \theta'_1) = L\left(\frac{2\pi}{3}\right) + L(-\pi) = a + b.$$

VI-VI)

Diagonal entry $j = i = 2$ (includes the image at $\theta'_2 = \pi$):

$$L_{22}^{(\text{half})} = L_0 + L(\theta_2 - \theta'_2) = L_0 + L(-\pi) = L_0 + b,$$

while $L_{11}^{(\text{half})} = L_{33}^{(\text{half})} = L_0$ because the corresponding images are at $\theta'_1 = 4\pi/3$ and $\theta'_3 = 2\pi/3$, which are separated from θ_1, θ_3 by $|\Delta| = 5\pi/3$ and $|\Delta| = \pi/3$, respectively, and their self-contributions are absorbed into the same local regularization constant at this order.

Collecting the entries (ordering $-\pi/3, 0, \pi/3$), we obtain

$$L^{(\text{half})} = \begin{pmatrix} L_0 & a & a+b \\ a & L_0+b & a \\ a+b & a & L_0 \end{pmatrix}. \quad (31)$$

The admissible subspace is set by the compartment volume constraint:

$$v_1 + v_2 + v_3 = 0.$$

A convenient basis for this 2D subspace is

$$w^{(1)} = (1, 0, -1), \quad w^{(2)} = (1, -2, 1).$$

Direct multiplication with (31) yields

$$L^{(\text{half})} w^{(1)} = \begin{pmatrix} L_0 - a - b \\ 0 \\ -(L_0 - a - b) \end{pmatrix} = (L_0 - a - b) w^{(1)}, \quad (32)$$

$$L^{(\text{half})} w^{(2)} = \begin{pmatrix} L_0 + b - a \\ -2(L_0 + b - a) \\ L_0 + b - a \end{pmatrix} = (L_0 + b - a) w^{(2)}. \quad (33)$$

Thus, on the admissible subspace, the eigenvalues are $L_0 - a - b$ and $L_0 + b - a$. Since

$$(L_0 + b - a) - (L_0 - a - b) = 2b > 0,$$

the larger eigenvalue (hence the fastest-growing admissible mode) corresponds to $w^{(2)} = (1, -2, 1)$. Therefore, the relevant eigenvalue of $L^{(\text{half})}$ is

$$\lambda_{L,2} = L_0 + \ln 2 - \ln(\sqrt{3}),$$

and the corresponding effective mass eigenvalue is

$$m_2 = \ell - \frac{h}{\pi} \lambda_{L,2} = \ell - \frac{h}{\pi} (L_0 + \ln 2 - \ln(\sqrt{3})). \quad (34)$$

VII)

From (18) we have

$$T_2^2 - T_1^2 = (\ln 2)^2 \frac{m_2 - m_1}{\Omega^2 R}.$$

Using (27)-(34),

$$m_2 - m_1 = -\frac{h}{\pi} [(L_0 + b - a) - (L_0 + 2a - b)] = \frac{h}{\pi} (3a - 2b) = \frac{h}{2\pi} \ln\left(\frac{27}{16}\right).$$

Both ℓ and L_0 cancel in the difference. With $h = d$, this gives

$$\boxed{T_2^2 - T_1^2 = \frac{(\ln 2)^2}{2\pi} \ln\left(\frac{27}{16}\right) \frac{d}{\Omega^2 R}.} \quad (35)$$

References

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