

## Updated Solution to Physics Cup Problem 2

To determine the condition for the formation of a multi-stream flow, we must compare the kinematics of wave breaking with the collisional relaxation of the gas.

### 1. Kinematic Wave Breaking

In the limit of negligible collisions (ballistic motion), particles move with constant velocity. A particle initially at position  $x_0$  moves to position  $x(t)$  given by:

$$x(t) = x_0 + v_x(x_0)t = x_0 + v_0 \sin(kx_0)t. \quad (1)$$

The flow becomes "multi-stream" when the mapping  $x_0 \rightarrow x(t)$  ceases to be monotonic, i.e., when particles from different initial locations arrive at the same final location. This "breaking" first occurs at the points of maximum negative velocity gradient. The breaking time scale is:

$$\tau_{break} \sim \frac{1}{kv_0}. \quad (2)$$

### 2. The Caustic Region

At the moment of breaking ( $t \approx \tau_{break}$ ), the linear dependence of position on the initial coordinate vanishes (a "fold catastrophe"). Expanding the trajectory near a node (shifted to  $x_0 = 0$  for convenience) where the gradient is maximal:

$$x(t) \approx x_0 (1 - kv_0 t) + \frac{1}{6} k^3 x_0^3 v_0 t. \quad (3)$$

At  $t = (kv_0)^{-1}$ , the linear term cancels, and the spreading is cubic:

$$x \sim k^2 x_0^3. \quad (4)$$

However, the gas has a finite temperature  $T$ . Thermal motion introduces a random displacement  $\delta x_{th} \sim v_{th} t \sim v_{th}/(kv_0)$ , where  $v_{th} \sim \sqrt{k_B T/m}$ . The "caustic region" consists of particles whose deterministic cubic displacement is comparable to or smaller than the thermal spread. The characteristic initial Lagrangian width  $x_0^*$  of the gas segment that collapses into this high-density region is found by equating the scales:

$$k^2 (x_0^*)^3 \sim \frac{v_{th}}{kv_0} \implies x_0^* \sim \frac{1}{k} \left( \frac{v_{th}}{v_0} \right)^{1/3}. \quad (5)$$

### 3. Transparency Condition

For a multi-stream flow to develop, the streams of particles must pass through this high-density caustic region without colliding. If they collide, the distribution thermalizes into a single-peaked shock wave instead.

The condition for transparency is that the optical depth through the caustic must be less than unity. The relevant quantity is the column density  $N_{col}$  (particles per unit area) integrated through the caustic. Since particle number is conserved, the column density is simply the initial density  $n$  multiplied by the Lagrangian width  $x_0^*$ :

$$N_{col} \sim nx_0^*. \quad (6)$$

The probability of collision  $P$  for a particle traversing this region is:

$$P \sim \sigma N_{col} \sim \sigma nx_0^*, \quad (7)$$

where  $\sigma \sim r^2$  is the collision cross-section. The condition for multi-streaming is  $P \lesssim 1$ :

$$\sigma n \frac{1}{k} \left( \frac{v_{th}}{v_0} \right)^{1/3} \lesssim 1. \quad (8)$$

#### 4. Result

Solving the inequality for  $v_0$ :

$$\left( \frac{v_{th}}{v_0} \right)^{1/3} \lesssim \frac{k}{n\sigma} \implies \frac{v_{th}}{v_0} \lesssim \left( \frac{k}{n\sigma} \right)^3. \quad (9)$$

Thus, the required condition for  $v_0$  is:

$$v_0 \gtrsim v_{th} \left( \frac{nr^2}{k} \right)^3. \quad (10)$$

Or, expressed in terms of the mean free path  $\lambda \sim (nr^2)^{-1}$ :

$$v_0 \gtrsim v_{th} \left( \frac{1}{k\lambda} \right)^3. \quad (11)$$

This cubic dependence arises because the density enhancement at the wave-breaking point significantly increases the collision rate compared to the unperturbed gas.