

Physics cup 2026 Problem 3

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Symmetries of the Circuit

To begin, we'll start by redrawing the circuit (see figure 1). In the given representation of the circuit there are nodes shorted to each other by wires, in the redrawing these redundancies will be removed, and the structure of the circuit might be a little clearer. This way the circuit and its symmetries are a little clearer and proceeding with the next steps will be easier, as we will soon see. In order for the symmetry of the circuit to be clearer, we will assume that the voltage at A is $+\frac{V_0}{2}$ and at B it's $-\frac{V_0}{2}$. Also, define the nodes C and D to be the midway nodes between A and B .

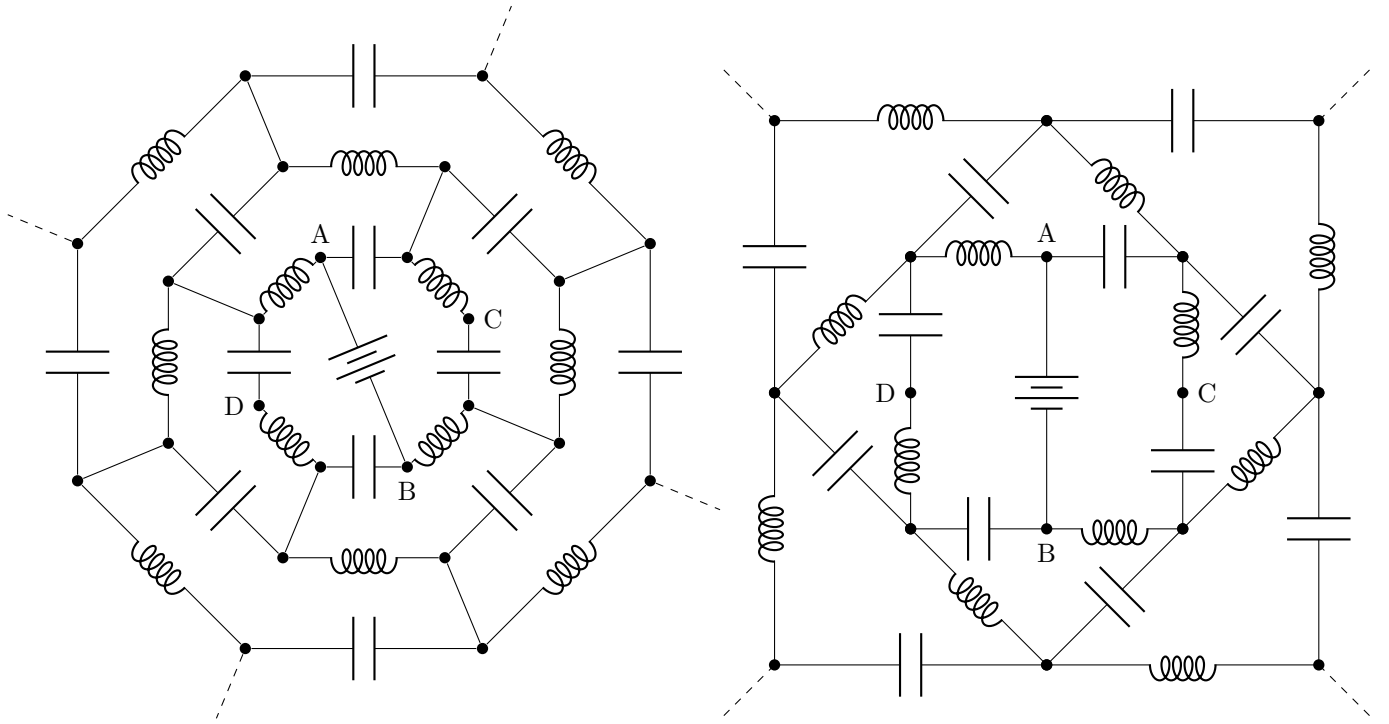


Figure 1: Two equivalent representations of the circuit. On the left is the original drawing and on the right a redrawing of it.

We will start with the simple symmetry - the 180° rotational symmetry (C_2 symmetry). Since rotating the circuit by 180° and switching the sign of the voltage returns the original circuit, then since everything is linear we know that voltages at opposing nodes are complements and the same thing is true for currents. This implies for example that $V_C = -V_D$.

Next, we will use the 90° rotational symmetry (C_4 symmetry). With this symmetry we will need to be a lot more careful since the input voltage does not obey this symmetry. Consider the connections between A and C and A and D . At first glance, they are similar but not identical since going from A to C there is a capacitor before the intersection and going from A to D there is an inductor before the intersection. Thus, we cannot just simply use the fact that they are symmetric. What is true though is that the connection from D to A is the same as the connection from A to C due to the 90° rotational symmetry. That is to say that the connection is exactly the same, but in the reverse order, which means the connections are in fact equivalent. Of course, the same is true for the connection between B and C, D . Thus points C and D are equivalent, meaning $V_C = V_D$. Using this along with the first symmetry we found, $V_C = -V_D$, we get:

$$V_C = V_D = 0 \quad (1)$$

This is a very powerful fact and will be very helpful for the rest of the solution. Note that unlike the first argument, this symmetry argument is very delicate, and does not hold for midway nodes in outer layers. For a demonstration of this fact, see the Extra Notes for a simulation result showing this.

Self Duality

Now consider the dual of the circuit (see Appendix). The nodes of the dual are obtained by placing a node of the dual inside each face of the circuit. All components separate two faces of the circuit, which in turn correspond to two nodes of the dual. In the dual circuit These two nodes are connected by the dual component of this component. The dual of a capacitor with capacitance C is an inductor with inductance C , the dual of an inductor is a capacitor with capacitance L and the dual of a voltage source V is a current source V . Note that the units of the dual differ from normal units, but we will soon normalize them. Construction of the dual:

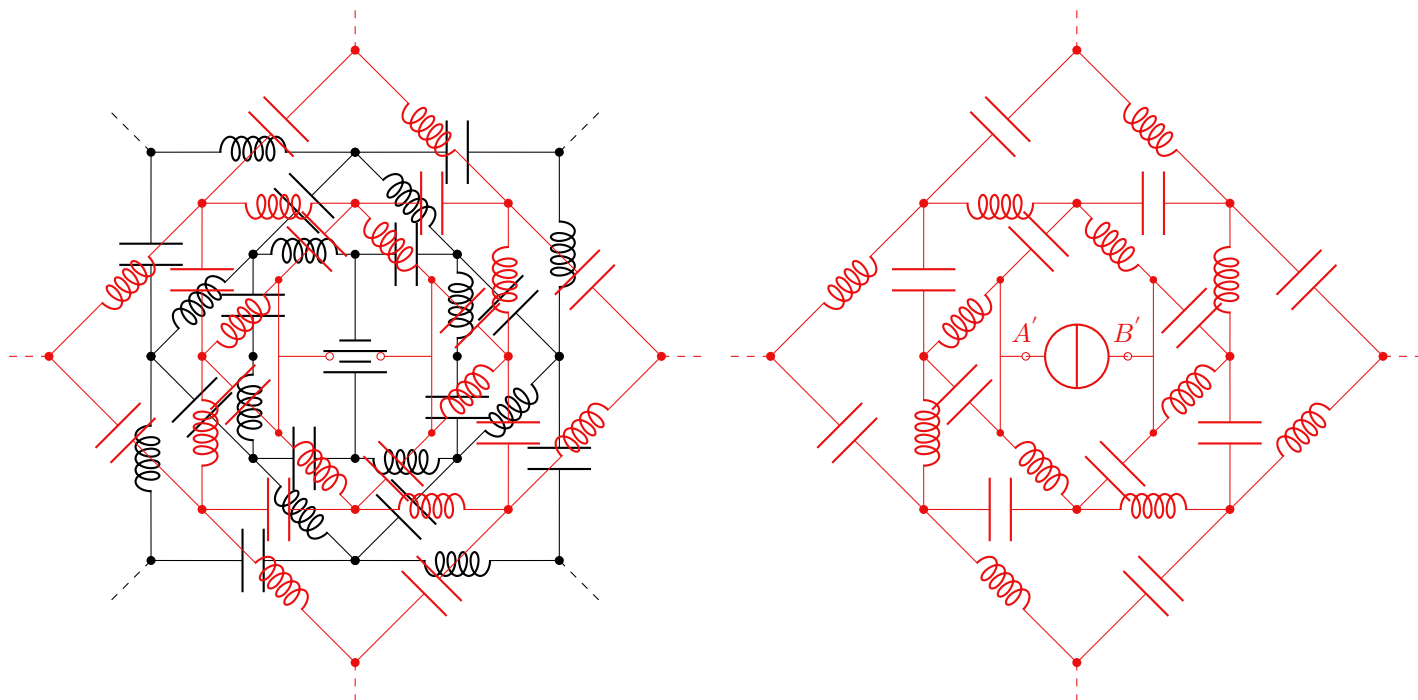


Figure 2: On the left: Construction of the dual circuit from the original. On the right: The dual circuit.

The important property of the dual is that its effective impedance is given by

$$Z^*(\omega) = \frac{1}{Z(\omega)} \quad (2)$$

Where $Z(\omega)$ is the effective impedance of the original circuit.

It will be easier to compare the two circuits when they have the same units. The dual impedances of the components are

$$Z_{capacitor} = \frac{1}{i\omega L}, Z_{inductor} = i\omega C$$

Thus if we multiply all impedances by $\frac{L}{C}$ we get the components to be the same as in the original circuit.

Notice that the dual is very similar to the original circuit - the original circuit is almost self dual. The only difference is that now nodes A, C and B, D are short-circuited (also there is a current source instead of a voltage source, but we will calculate the impedance between the points connected by the source, so it doesn't matter). Now, consider the original circuit superposed with a 90° rotated version of itself. Recall from (1) that $V_C = V_D = 0$ in the original circuit. Consequently, in the superposition the voltages at A and C are $+\frac{V_0}{2}$ while at B and D they are $-\frac{V_0}{2}$. This is now exactly the same as the dual circuit. Notice that the input current is twice larger in this superposed circuit since it is the superposition of two identical copies of the original. Meanwhile, the input voltage didn't change. Thus:

$$Z^*(\omega) \cdot \frac{L}{C} = \frac{Z(\omega)}{2}$$

Plugging in (2):

$$\frac{1}{Z(\omega)} \cdot \frac{L}{C} = \frac{Z(\omega)}{2}$$

$$Z(\omega) = \sqrt{\frac{2L}{C}}$$

Since the impedance is constant, the whole circuit is equivalent to a resistor $R = \sqrt{\frac{2L}{C}}$ between A and B . Thus:

$$\boxed{I(t) = \frac{V(t)}{Z} = V_0 \sqrt{\frac{C}{2L}}}$$

Extra Notes

1. Had $Z(\omega)$ not been constant, finding $I(t)$ would have been tougher, but still there is a simple procedure to calculate it. Using superposition we can decompose the signal $V(t)$ (which is a step function in this case) into a superposition of pure oscillating signals, i.e. $V(t) = \int_{-\infty}^{\infty} V_0(\omega) e^{i\omega t} d\omega$. This can be done using the Fourier transform. Then, for each of these signals the current response is simply $I(\omega) = \frac{V_0(\omega)}{Z(\omega)}$. Finally to get $I(t)$, we have to use the inverse transform on $I(\omega)$.
2. To check the result and demonstrate both the symmetry between C and D and how that argument does not trivially extend to outer layers, I wrote python code to simulate the circuit with a given amount of layers. In order to ensure numerical stability, the voltage was increased over a short time period at the start, so the quick rise at the start of the graphs is a result of that. The values used for the simulation were $L = 1H, C = 1F, V_0 = 1V$. Thus the expected input current is $I(t) = \frac{1}{\sqrt{2}}A \approx 0.7A$. The simulation seems to match that pretty well. Note that only in layer 0 are the midway nodes at $V = 0$, demonstrating the delicacy of the argument for layer 0 and how nontrivial it is that $V_C = V_D$.

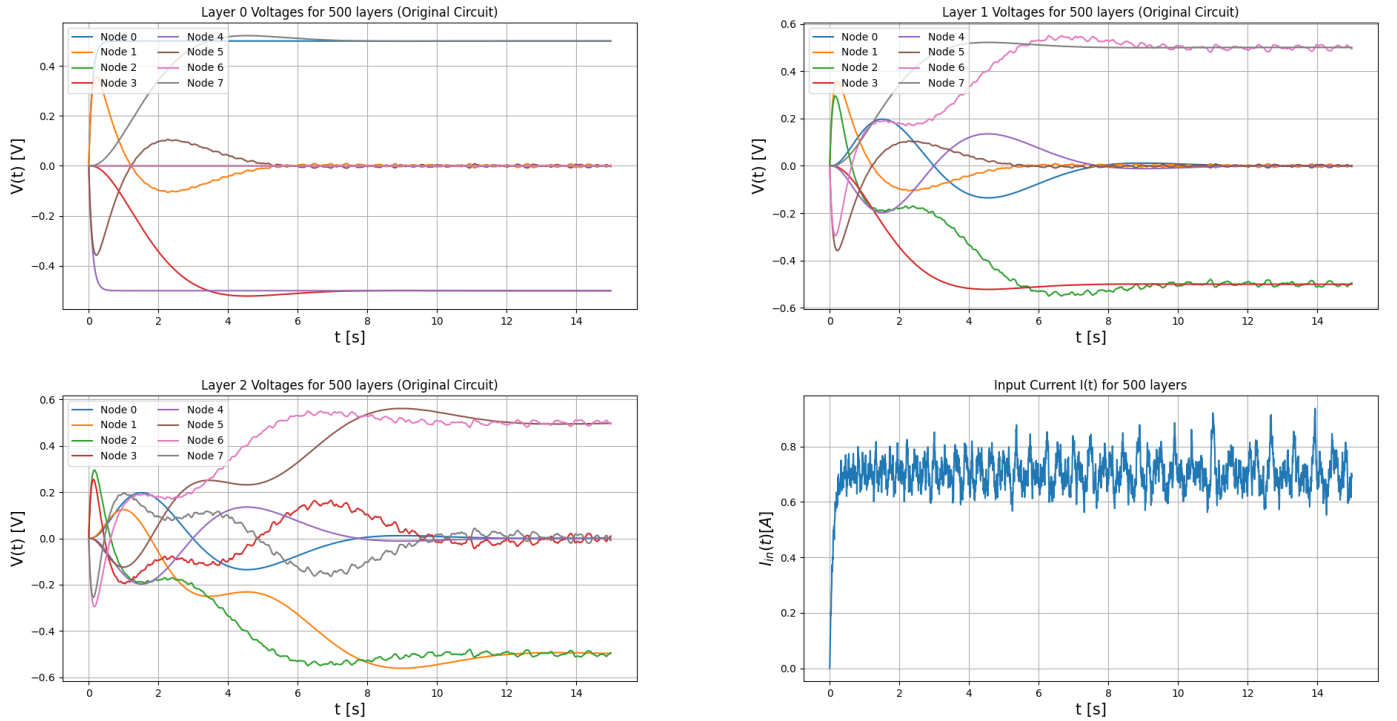


Figure 3: Results of the python simulation for 500 total layers ran for 15s and using $L = 1H, C = 1F, V_0 = 1V$. The graphs are of the voltages in layers 0, 1, 2 and of the input current against time. Nodes 0, 2, 4, 6 correspond to A, C, B, D respectively. For outer layers - they share nodes with an inner layer and have consistent numbering.

Appendix: Dual Circuits

We will start by considering circuits only including voltages sources and resistors, then generalize the construction for capacitors and conductors as well using impedances.

The concept of a dual circuit comes from the duality between I, V in all circuit equations. First in Ohm's law they are symmetric, and also in KCL and KVL have the same form - KVL says the sum of voltage differences in a loop is 0, and KCL says the sum of currents going into a node is 0. Thus, if we build a circuit where nodes and wires switch roles, then so do I and V , on the condition that we replace the resistances R by $\frac{1}{R}$.

In the case of planar circuits, i.e. circuits that can be drawn without intersecting wires, this is not too difficult. Note that outer face of the circuit, that reaches infinity is also considered a face and is considered to be enclosed by all the wires it touches. From KCL, the sum of currents at any node (not connected to a source) is zero. As a result, currents in loops are consistent, meaning each loop can have a current attributed to it. A wire, of course, is part of 2 loops, so the current in it is the sum of the current from the 2 loops (with the appropriate signs).

Say a loop i has current I_i (anti-clockwise). Thus, we can consider the face encircled by the loop to have current I_i . So, we can put a node inside each face and each node now has a current associated to it. Note that this is due to KCL, but now a node having a defined current is like KVL, where each nodes has a defined voltage. Now, connect all nodes through the resistors separating their faces. Suppose Two nodes have currents I_i, I_j and the resistor between their faces is R_{ij} . Then the current in the wire is $I_i - I_j$ and the voltage on the resistor is $V_{ij} = R_{ij} (I_i - I_j)$. Now KVL in the original circuit is that the sum of voltage differences in a loop is 0. In the dual circuit, it means that the sum of voltages on all components in the loop is 0, which means the sum of voltages due to the connection of the node of the loop to all the others is 0. Notice this means KVL in the original circuit became KCL in the dual.

Now I, V have switched places: There is a current associated with each node and a voltage associated with each wire. Thus we can consider I to be a voltage and V to be a current in the dual. Thus to keep Ohm's law, we now have:

$$\Delta I = I_i - I_j = \frac{V_{ij}}{R_{ij}}$$

This means the new resistances are $R_{ij}^* = \frac{1}{R_{ij}}$. Of course the voltage source now has to be swapped with a current source and vice versa. The effective resistance between the two points connected by the source will now then be:

$$R_{eff}^* = \frac{1}{R_{eff}}$$

Note that the outer face of the circuit (reaching infinity) is also a face, and should have a node associated with it.

Also note that the dual transformation inverts the dimensions of impedance, meaning the dual circuit units differ from the original circuit units by a factor of the units of Resistance squared. To compare the dual directly to the primal, it is convenient to normalize these units by multiplying the dual impedances by a characteristic resistance factor R_{char}^2 . This restores the units of the dual to be the same as the original.

As for inductors and capacitors, as explained in the extra note, it is enough to show the duality for AC circuits, since any signal can be decomposed into a superposition of these. Thus we can just treat them the same as resistors:

$$Z_C = \frac{1}{i\omega C}, Z_L = i\omega L$$

$$\Rightarrow Z_C^* = i\omega C, Z_L^* = \frac{1}{i\omega L}$$

So, we see that the dual of a capacitor with capacitance C is an inductor of inductance C and the dual of an inductor of inductance L is a capacitor with capacitance L . Also the effective impedance of the circuit is now found instead of the resistance:

$$Z_{eff}^* = \frac{1}{Z_{eff}}$$