

# Physics cup 2026 Problem 1

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## 1 Preliminary Considerations

This problem is almost purely mathematical. Thus, we will start with some physical observations and turn the problem into a mathematical one. Since the brick is squeezed between the planes in zero gravity, the normal force exerted on it from both planes is equal. The friction between the block and both planes is dry and has the same coefficient of kinetic friction, so both planes will exert the same force on the block in the direction of the relative velocity vector. Denote the magnitude of the acceleration due to each plane separately  $a_1$ . Note that  $a_1 \neq a_0$  and working with  $a_1$  will be a lot easier, so throughout the solution we will use that and plug in the relationship between  $a_1$  and  $a_0$  in the end.

Now let's introduce dimensionless variables:

$$\tau = \frac{a_1}{u_0} t, x(\tau) = \frac{u_x(t)}{u_0}, y(\tau) = \frac{u_y(t)}{u_0}, r(\tau) = \frac{u(t)}{u_0}, \lambda = \frac{v}{u_0}$$

$x, y, r$  are dimensionless velocities and should not be confused with the positions of the block. The only distance in this problem is  $l_y$ , so we can use this notation since it can't be confused with the position in the context of this problem.

The initial conditions are

$$x(0) = 0, y(0) = 1, r(0) = 1$$

$l_y$  can only be dependent on  $u_0, a_1, \lambda$  and thus from dimensional analysis we obtain:

$$l_y = \frac{u_0^2}{a_1} f(\lambda)$$

We will now be interested in finding  $f(\lambda)$  which is given by

$$f(\lambda) = \frac{a_1 l_y}{u_0^2} = \frac{a_1}{u_0^2} \int_0^\infty u_y dt = \int_0^\infty y d\tau$$

## 2 The Conserved quantity

Consider the problem in velocity space. Each plane contributes an acceleration (which in velocity space is a velocity) of the same magnitude in the direction of the relative velocity. This acceleration is independent of the distance between  $\vec{0}, \vec{v}$  and  $\vec{u}$ . So from symmetry,  $|\vec{u}|$  and  $|\vec{v} - \vec{u}|$  will shrink at the same rate. Thus  $|\vec{v} - \vec{u}| - |\vec{u}| = \text{const.}$

We can also see this mathematically: denote  $\vec{w} = \vec{u} - \vec{v}$ . Then

$$\begin{aligned} \vec{a} &= -a_1 (\hat{u} + \hat{w}) \\ \frac{d}{dt} |\vec{u}| &= \vec{a} \cdot \hat{u} = -a_1 (1 + \hat{w} \cdot \hat{u}) \end{aligned}$$

$$\frac{d}{dt}|\vec{w}| = \vec{a} \cdot \hat{w} = -a_1 (\hat{u} \cdot \hat{w} + 1) = \frac{d}{dt}|\vec{u}|$$

$$\implies |\vec{v} - \vec{u}| - |\vec{u}| = |\vec{w}| - |\vec{u}| = \text{const}$$

Or equivalently in the dimensionless variables:

$$|\vec{\lambda} - \vec{r}| - |\vec{r}| = \sqrt{(\lambda - x)^2 + y^2} - r = \text{const}$$

Notice that this is the definition of a hyperbola. So the block's trajectory in velocity space will be a hyperbola with foci at  $(0,0), (\lambda,0)$  that goes through  $(0,1)$ . It of course won't be the whole hyperbola, just the part between  $(0,1)$  and the  $x$ -axis, because on the  $x$ -axis the acceleration is 0 (of course only between the foci).

Having derived this relationship involving  $x, y$ , we can now relate any two of  $x, y, r$ . We will now derive a relationship between  $y, r$  that will be useful later. First of all though, we should define the constant of motion:

$$\mu \equiv \sqrt{1 + \lambda^2} - 1 = \sqrt{(\lambda - x)^2 + y^2} - r = \text{const} \quad (1)$$

To begin, we will relate  $x, r$ :

$$\begin{aligned} \sqrt{r^2 + \lambda^2 - 2\lambda x} - r &= \sqrt{1 + \lambda^2} - 1 = \mu \\ r^2 + \lambda^2 - 2\lambda x &= r^2 + 2\mu r + \mu^2 \\ x &= -\frac{1}{2\lambda} (2\mu r + \mu^2 - \lambda^2) = \frac{\mu}{\lambda} (1 - r) \end{aligned}$$

For the final step, the following identity was used:

$$\begin{aligned} \lambda^2 - \mu^2 &= \lambda^2 - \left( \sqrt{1 + \lambda^2} - 1 \right)^2 = \lambda^2 - \left( 2 + \lambda^2 - 2\sqrt{1 + \lambda^2} \right) = 2 \left( \sqrt{1 + \lambda^2} - 1 \right) = 2\mu \\ \implies \lambda^2 - \mu^2 &= 2\mu \end{aligned} \quad (2)$$

We will now continue in order to get a relationship involving  $y, r$ :

$$\begin{aligned} y^2 &= r^2 - x^2 = r^2 - \frac{\mu^2}{\lambda^2} (r^2 - 2r + 1) = \frac{1}{\lambda^2} ((\lambda^2 - \mu^2) r^2 + 2\mu^2 r - \mu^2) \\ &\stackrel{2}{=} \frac{\mu}{2\lambda^2} (4r^2 + 4\mu r - 2\mu) = \frac{\mu}{2\lambda^2} ((2r + \mu)^2 - 2\mu - \mu^2) \stackrel{2}{=} \frac{\mu}{2\lambda^2} ((2r + \mu)^2 - \lambda^2) \\ \implies 2r + \mu &= \sqrt{\left( \frac{2\lambda^2}{\mu} \right) y^2 + \lambda^2} = \lambda \sqrt{\frac{2y^2}{\mu} + 1} \end{aligned} \quad (3)$$

### 3 The EOMs and final integration

Now All that is left to do is to calculate:

$$f(\lambda) = \int_0^\infty y d\tau = \int_{y(0)}^{y(\infty)} y \cdot \frac{d\tau}{dy} dy = - \int_0^1 y \cdot \frac{d\tau}{dy} dy$$

So we need to find  $\frac{d\tau}{dy}$  which we can get from finding  $\frac{dy}{d\tau}$ , which is easy due to it being the acceleration in the  $y$ -axis. Using the conservation law, the block's velocity relative to the resting and moving plane is  $r, r + \mu$  respectively, and the  $y$ -components of both is  $y$ . So

$$\frac{dy}{d\tau} = \frac{1}{a_1} \frac{du_y}{dt} = - \left[ \frac{y}{r} + \frac{y}{r+\mu} \right] = -y \cdot \frac{2r+\mu}{r(r+\mu)} = -4y \frac{2r+\mu}{(2r+\mu)^2 - \mu^2}$$

We now just have to integrate:

$$\begin{aligned} f(\lambda) &= \int_1^0 y \cdot \frac{d\tau}{dy} dy = \int_0^1 y \cdot \frac{(2r+\mu)^2 - \mu^2}{4y \cdot (2r+\mu)} dy = \frac{1}{4} \int_0^1 \left( 2r+\mu - \frac{\mu^2}{2r+\mu} \right) dy \\ &\stackrel{3}{=} \frac{1}{4} \int_0^1 \left( \lambda \sqrt{\frac{2y^2}{\mu} + 1} - \frac{\mu^2}{\lambda \sqrt{\frac{2y^2}{\mu} + 1}} \right) dy = \frac{1}{4} \sqrt{\frac{\mu}{2}} \int_0^{\sqrt{\frac{2}{\mu}}} \left( \lambda \sqrt{z^2 + 1} - \frac{\mu^2}{\lambda \sqrt{z^2 + 1}} \right) dz \end{aligned}$$

These integrals are easily evaluated using the hyperbolic substitution  $z = \sinh(\xi)$ :

$$\begin{aligned} \int \sqrt{z^2 + 1} dz &= \frac{1}{2} \left( \sinh^{-1}(z) + z \sqrt{z^2 + 1} \right) + C \\ \int \frac{1}{\sqrt{z^2 + 1}} dz &= \sinh^{-1}(z) + C \\ \Rightarrow f(\lambda) &= \frac{1}{4} \sqrt{\frac{\mu}{2}} \left[ \frac{(\lambda^2 - 2\mu^2) \sinh^{-1}(z) + \lambda^2 z \sqrt{z^2 + 1}}{2\lambda} \right]_0^{\sqrt{\frac{2}{\mu}}} \\ &= \frac{1}{8} \left[ \sqrt{\frac{\mu}{2}} \cdot \frac{(\lambda^2 - 2\mu^2)}{\lambda} \sinh^{-1} \left( \sqrt{\frac{2}{\mu}} \right) + \lambda \sqrt{\frac{2}{\mu} + 1} \right] \end{aligned}$$

Now, trying to get this equation in terms of only  $\lambda$  will be terrible because  $\mu(\lambda)$  isn't very nice (and the same is true for the other way around), so let's introduce a new variable that is close to both:  $s = \sqrt{1 + \lambda^2} = \mu + 1$ .

$$\begin{aligned} f(\lambda) &= \frac{1}{8} \left[ \sqrt{\frac{s-1}{2}} \cdot \frac{(s^2 - 1 - 2(s-1)^2) \sinh^{-1} \left( \sqrt{\frac{2}{s-1}} \right)}{\sqrt{s^2 - 1}} + \sqrt{s^2 - 1} \sqrt{\frac{s+1}{s-1}} \right] \\ &= \frac{1}{8} \left[ \frac{(-s^2 + 4s - 3) \sinh^{-1} \left( \sqrt{\frac{2}{s-1}} \right)}{\sqrt{2}(s+1)} + s + 1 \right] \\ &= \frac{s+1}{8} + \frac{(s-1)(3-s)}{16} \sqrt{\frac{2}{s+1}} \sinh^{-1} \left( \sqrt{\frac{2}{s-1}} \right) \\ \Rightarrow f(\lambda) &= \frac{s+1}{8} + \frac{(s-1)(3-s)}{16} \sqrt{\frac{2}{s+1}} \sinh^{-1} \left( \sqrt{\frac{2}{s-1}} \right); s = \sqrt{1 + \lambda^2} \end{aligned} \quad (4)$$

And it's not likely going to get much more compact than this. We have gotten  $l_y$  in terms of  $a_1$ , but were asked to express the final result in terms of  $a_0$ , so next, we will relate it to  $a_1$ .

$$\begin{aligned} \vec{a}_0 &= -a_1 (\hat{u} + \hat{w}) = -a_1 \left( \left( 1 + \frac{1}{\sqrt{1+\lambda^2}} \right) \hat{y} + \frac{\lambda}{\sqrt{1+\lambda^2}} \right) \\ \left( \frac{a_0}{a_1} \right)^2 &= \frac{|\vec{a}_0|^2}{a_1^2} = \left( 1 + \frac{1}{s} \right)^2 + \left( \frac{\sqrt{s^2 - 1}}{s} \right)^2 = \frac{(s+1)^2 + s^2 - 1}{s^2} = \frac{2s^2 + 2s}{s^2} = 2 \frac{s+1}{s} \end{aligned}$$

So we finally get

$$l_y = \frac{u_0^2}{a_1} f(\lambda) = \frac{u_0^2}{a_0} \sqrt{2 \frac{s+1}{s}} f(\lambda)$$

Plugging in 4, we get the result:

$$\begin{aligned} l_y &= \frac{u_0^2}{a_0} \sqrt{2 \frac{s+1}{s}} \left( \frac{s+1}{8} + \frac{(s-1)(3-s)}{16} \sqrt{\frac{2}{s+1}} \sinh^{-1} \left( \sqrt{\frac{2}{s-1}} \right) \right) \\ &= \frac{u_0^2}{a_0} \left( \frac{1}{8} \sqrt{2 \frac{(s+1)^3}{s}} + \frac{(s-1)(3-s)}{8\sqrt{s}} \sinh^{-1} \left( \sqrt{\frac{2}{s-1}} \right) \right) \end{aligned}$$

$$l_y = \frac{u_0^2}{8a_0} \left( \sqrt{2 \frac{(s+1)^3}{s}} + \frac{(s-1)(3-s)}{\sqrt{s}} \sinh^{-1} \left( \sqrt{\frac{2}{s-1}} \right) \right); s = \sqrt{1 + \left( \frac{v}{u_0} \right)^2}$$

We are asked to find  $l_y$  for  $u_0 = v = 1 \frac{m}{s}$ ,  $a_0 = 1 \frac{m}{s^2}$ . So  $\lambda = 1$ ,  $s = \sqrt{2}$  and the  $y$ -displacement evaluates to:

$$l_y \approx 0.6631m$$

## 4 Sanity checks

In the case  $v = 0$ , the acceleration is constant and in the negative  $y$  direction, so we expect  $l_y = \frac{u_0^2}{2a_0}$ . Plugging in  $v = 0$  we get  $s = 1$  and it's pretty easy to see that the second term goes to zero since  $\sinh^{-1}$  grows logarithmically, so we are left with only the first term which evaluates to  $l_y = \frac{u_0^2}{8a_0} (\sqrt{16} + 0) = \frac{u_0^2}{2a_0}$ , as expected.

In the case  $v \rightarrow \infty$  we expect a finite displacement and also  $l_y > \frac{u_0^2}{2a_1} \iff f(\lambda) > \frac{1}{2}$  since the acceleration in the  $y$ -axis is always smaller than  $a_1$ . At first glance it looks like  $f$  diverges because both terms diverge, but they in fact cancel out. A more careful analysis to first order in  $\frac{1}{s}$  yields the following:

$$\begin{aligned} f(s \rightarrow \infty) &= \frac{s+1}{8} + \frac{(s-1)(3-s)}{16} \sqrt{\frac{2}{s+1}} \sinh^{-1} \left( \sqrt{\frac{2}{s-1}} \right) \\ &\approx \frac{s+1}{8} + \frac{-s^2+4s}{16} \sqrt{\frac{2}{s+1}} \cdot \sqrt{\frac{2}{s-1}} \left( 1 - \frac{1}{6} \cdot \frac{2}{s-1} \right) \\ &\approx \frac{s+1}{8} + \frac{-s+4}{8} \sqrt{\frac{1}{1+s^{-2}}} \left( 1 - \frac{1}{3s} \right) \approx \frac{s+1}{8} + \frac{-s+4}{8} + \frac{1}{24} = \frac{2}{3} > \frac{1}{2} \end{aligned}$$