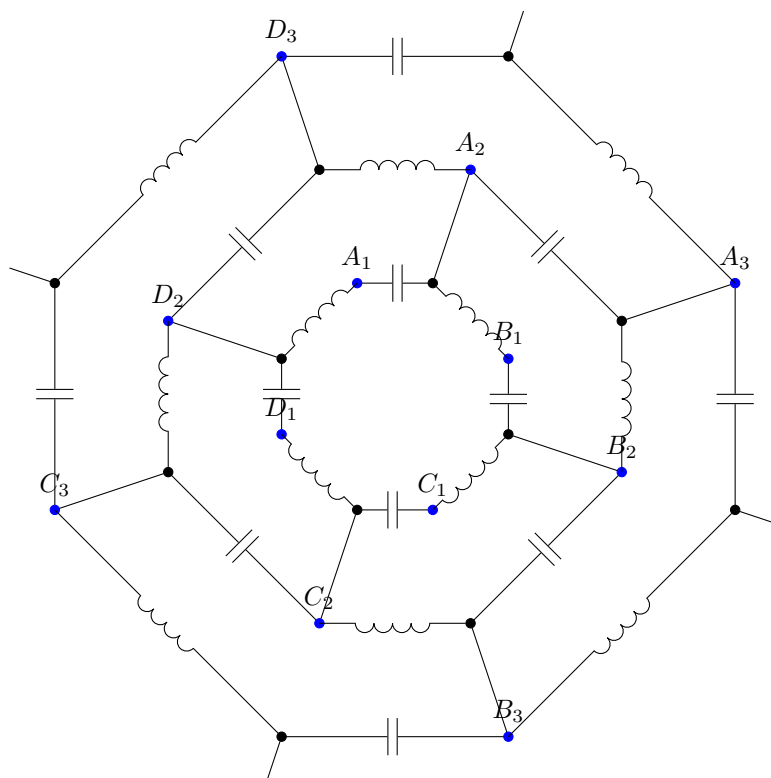


Physics Cup. Problem 3

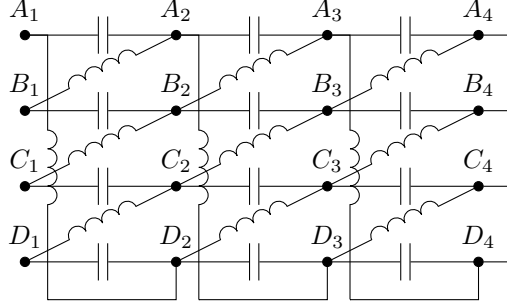
Artyom Eryomin

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1 Simplifying the circuit



Let us rename all the nodes of the circuit — A_1 , B_1 , C_1 , D_1 , A_2 , B_2 , etc., as it drawn on the circuit. Voltage is applied to A_1 and C_1 . Thus we can redraw the circuit:



This circuit has infinitely many cascades with 4 contacts. Now it looks much simpler.

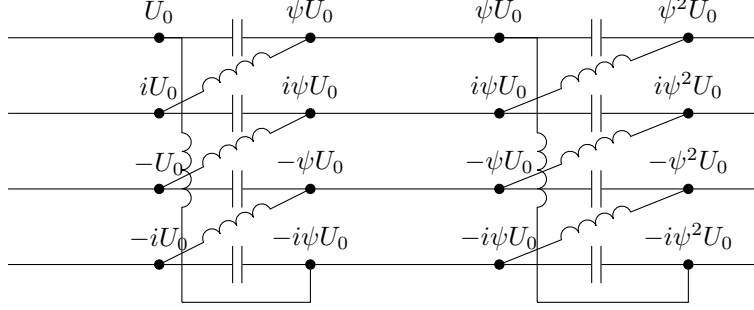
2 Calculating impedance on angular frequency

ω

2.1 A particular case of connection

Now we consider what happens when we apply potential with complex amplitude U_0 to A_1 , iU_0 to B_1 , $-U_0$ to C_1 and $-iU_0$ to D_1 . Suppose that current I_0 flows into A_1 . Due to the symmetry of rotation, currents iI_0 , $-I_0$ and $-iI_0$ flow into B_1 , C_1 , D_1 respectively (here and there we consider only complex amplitudes, and ignore $e^{i\omega t}$).

Let potential of A_2 is ψU_0 (it is proportional to U_0 , ψ is some complex number). Due to the symmetry of rotation, potentials of B_2 , C_2 and D_2 are $i\psi U_0$, $-\psi U_0$, $-i\psi U_0$. But if we consider all the cascades from the second, we get the same circuit, and currents through A_2 , B_2 , C_2 , D_2 are ψI_0 , $i\psi I_0$, $-\psi I_0$, $-i\psi I_0$ respectively. We can continue thinking this way on the next cascades, and we get that potential on A_i is $\psi^{i-1} U_0$, and current is $\psi^{i-1} I_0$. It is consequence of the recursive structure of the circuit and its rotation symmetry. We can show currents and potentials on the circuit:



(currents flowing through the wires are $I_0, \psi I_0, \psi^2 I_0, iI_0, i\psi I_0, i\psi^2 I_0$, etc.). We can write the first Kirchhoff's law in nodes. We need only two equations (all the rest equations differs only on multiplication on $\psi^k, i\psi^k, -\psi^k$ and $-i\psi^k$):

$$I_0 = \frac{U_0 - \psi U_0}{r} + \frac{U_0 + i\psi U_0}{R}$$

$$\psi I_0 = \frac{U_0 - \psi U_0}{r} + \frac{iU_0 - \psi U_0}{R}$$

($r \equiv \frac{1}{i\omega C}$, $R \equiv i\omega L$ — impedances of capacitor and inductor), and solve them:

$$\psi \left(\frac{1 - \psi}{r} + \frac{1 + i\psi}{R} \right) = \frac{1 - \psi}{r} + \frac{i - \psi}{R}$$

$$R\psi(1 - \psi) + r\psi(1 + i\psi) = R(1 - \psi) + r(i - \psi)$$

$$(ir - R)\psi^2 + (R + r + R + r)\psi - (ir + R) = 0$$

$$\psi = \frac{-(R + r) \pm \sqrt{(R + r)^2 + (ir + R)(ir - R)}}{ir - R} = \frac{-(R + r) \pm \sqrt{R^2 + 2Rr + r^2 - r^2 - R^2}}{ir - R}$$

$$\psi = \frac{-(R + r) \pm \sqrt{2Rr}}{ir - R}$$

Now we can find I_0 :

$$I_0 = U_0 \frac{1 - \psi}{r} + U_0 \frac{1 + i\psi}{R} = U_0 \frac{R(ir - R + R + r \mp \sqrt{2Rr}) + r(ir - R - iR - ir \pm i\sqrt{2Rr})}{Rr(ir - R)}$$

$$I_0 = U_0 \frac{Rr(i + 1 - 1 - i) \pm (ir - R)\sqrt{2Rr}}{Rr(ir - R)} = \pm U_0 \sqrt{\frac{2}{Rr}}$$

2.2 Connection in the initial task

Now we can use current superposition. If we apply currents $I_0, iI_0, -I_0, -iI_0$, we get some potentials $\pm U_0$. But if we apply currents $I_0, -iI_0, -I_0, iI_0$ (in the opposite order), we get the same circuit with replacement capacitors on

inductors and inductors on capacitors. But potentials don't change if we replace R and r as we can see in the previous formula. Then, in the second connection we got $\pm U_0$ again.

And we need to add this two connections — then currents in B_1 and D_1 become zero, but in A_1 and $C_1 = 2I_0$, and potentials are $\pm 2U_0$. So, the voltage is $4U_0$, and the equivalent impedance between A_1 and C_1 is

$$Z = \frac{4U_0}{2I_0} = 2\sqrt{\frac{Rr}{2}} = \sqrt{2Rr}$$

3 The final answer

For each angular frequency ω , the equivalent impedance is

$$Z = \sqrt{2Rr} = \sqrt{\frac{2i\omega L}{i\omega C}} = \sqrt{\frac{2L}{C}}$$

And it doesn't depend on frequency! The signal $V(t) = V_0$ for $t > 0$ and 0 for $t < 0$ can be represented as $-\frac{V_0}{2} + \frac{V_0}{2}\text{sgn}(t)$, and we can decompose it on periodic signal by the Fourier transform (for $\text{sgn}(t)$ it is $\frac{2}{i\omega}$, but we actually don't need it). And for each signal, the impedance is $\sqrt{\frac{2L}{C}}$. Then the circuits behaves just like a resistor with resistance $\sqrt{2L/C}$, so the current will jump instantly, and it's value is $U_0\sqrt{\frac{C}{2L}}$. So, the answer is

$$I(t) = \begin{cases} 0, & t < 0 \\ U_0\sqrt{\frac{C}{2L}}, & t \geq 0 \end{cases}$$