

Physics Cup 2026 Problem 4

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Inspiration for the Solution

We will start by getting some intuition of the water flow in both cases. In the holes, the surface tension acting on the $d \times d \times d$ cube inside is of order σd , while the centrifugal force acting on it is on order $\rho d^3 R \Omega^2$, which is much less than the surface tension thanks to the problem statement's assumption that $\sigma \gg \rho d^2 R \Omega^2$. This means that when the plugs in the holes are removed, surface tension prevents the water from flying out of the holes immediately. Therefore, for the described motion of water to occur, we will have to displace the water in the holes first. We will assume the water is incompressible, such that if we displace the water in one hole, that displaced water must come out of the other 5 holes.

We will work in the frame rotating with the container. The fluid is assumed to be inviscid and is initially irrotational, and there are no dissipative forces present in the system, meaning that the fluid must always be irrotational. Combining this with the incompressibility condition gives us

$$\nabla \cdot \mathbf{v} = 0, \nabla \times \mathbf{v} = \mathbf{0}$$

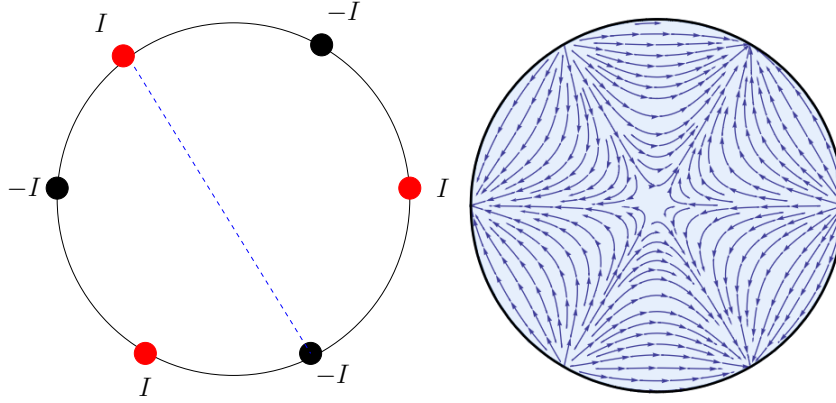
These equations resemble the divergence-free and curl-free current density $\mathbf{J}$ in circuit problems. This gives us an idea to consider an equivalent problem of current. The reason for choosing current specifically is this: The water's kinetic energy is $K = \int \frac{1}{2} \rho v^2 dV$, where we are integrating over the whole volume of water. The total power dissipated in a circuit has a similar form: $P = \int \rho_{res} J^2 dV$. We can, however, just find the power by multiplying the effective input current with the voltage drop between a current source and a current sink. Then, if one makes the mapping $P \leftrightarrow K$, $\rho_{res} \leftrightarrow \frac{1}{2} \rho$, $\mathbf{J} \leftrightarrow \mathbf{v}$, the total kinetic energy of the water can be recovered.

To construct our circuit, we will place small contacts where the holes are to inject and extract current, representing the displacement of water in the holes. The exact shape of the contacts does not make a significant difference to the current far away from the holes (which is where the vast majority of the dissipated power is), so we can just assume they are circular with diameter d , representing

the hole which is a gap of side length d . Since the container thickness $d \ll R$ this is effectively a two-dimensional problem, so we can superpose solutions of current spreading out radially from each current source.

Case 1: Without the Partition

This solution must be symmetric about the centre of the disk and it must be symmetric/antisymmetric about a rotation of the disk by a multiple of $\frac{\pi}{3}$, because one cannot distinguish such a rotated disk from an unrotated disk. We must also keep in mind the boundary condition that the current at the edge of the disk must flow parallel to it i.e. their equipotential lines must be normal to the disk. This motivates the solution on the left in the below figure, comprised of three current sources in an equilateral triangle, and three current sinks in another equilateral triangle. The resulting streamlines are shown on the right.



For a single source, the current spreads out radially, such that $J = \frac{I}{2\pi r d}$. Since $\mathbf{E} = \rho_{res} \mathbf{J}$, we obtain a potential $V = \frac{\rho_{res} I}{2\pi d} \ln \frac{r}{r_0}$, where r_0 is an arbitrary reference point. Therefore, the potential difference between two points a distance r_1 and r_2 away is $\Delta V = \frac{\rho_{res} I}{2\pi d} \ln \frac{r_2}{r_1}$. Now let's use this to find the potential difference between any current source and any current sink (it doesn't matter which ones we choose because all current sources are at the same potential, and likewise for the current sinks). For simplicity let's take any two diametrically opposite points (the dashed line in the figure). Then, going clockwise from 11 o'clock and summing each element's contribution to the potential, we have

$$\Delta V \approx \frac{\rho_{res} I}{2\pi d} \left(\ln \frac{2R}{\frac{d}{2}} - \ln \sqrt{3} + \ln \frac{1}{\sqrt{3}} - \ln \frac{\frac{d}{2}}{2R} + \ln \frac{1}{\sqrt{3}} - \ln \sqrt{3} \right)$$

$$\Delta V \approx \frac{\rho_{res} I}{2\pi d} \ln \frac{16R^2}{9d^2}$$

Note that the effective current is not $3I$ but rather $\frac{3}{2}I$. This is because only half of each current source's/sink's field lines are inside the disk. Thus

$$P = \frac{3}{2}I\Delta V = \frac{3\rho_{res}I^2}{4\pi d} \ln \frac{16R^2}{9d^2}$$

Using the mapping $P \leftrightarrow K$, $\rho_{res} \leftrightarrow \frac{1}{2}\rho$, $I \leftrightarrow 2vd^2$ (the total velocity flux in each hole is vd^2 , which corresponds to half of the source/sink current, $\frac{I}{2}$, as mentioned just above), we get

$$K = \frac{3\rho d^3}{2\pi} \left(\ln \frac{16R^2}{9d^2} \right) v^2$$

The centrifugal potential energy U decreases when water moves from one hole to another. In this case, we can imagine three $d \times d \times x$ cuboids of water with centre of mass at $r = R + d - \frac{x}{2}$ disappearing from the "current source" holes, and three $d \times d \times x$ cuboids of water with centre of mass at $r = R + d + \frac{x}{2}$ appearing in the "current sink" holes. Then the centrifugal potential energy is

$$U = 3 \left(-\frac{1}{2} (\rho x d^2) \Omega^2 \left(R + d + \frac{x}{2} \right)^2 + \frac{1}{2} (\rho x d^2) \Omega^2 \left(R + d - \frac{x}{2} \right)^2 \right)$$

$$U \approx \frac{3}{2} \rho x \Omega^2 d^2 (-2x(R+d)) \approx -3\rho \Omega^2 R d^2 x^2$$

Now since $v = \dot{x}$, we invoke energy conservation and demand that $K + U = 0$:

$$\frac{3\rho d^3}{2\pi} \left(\ln \frac{16R^2}{9d^2} \right) \dot{x}^2 - 3\rho \Omega^2 R d^2 x^2 = 0$$

$$\dot{x} = \sqrt{\frac{2\pi\Omega^2 R}{d \ln \left(\frac{16R^2}{9d^2} \right)}} x$$

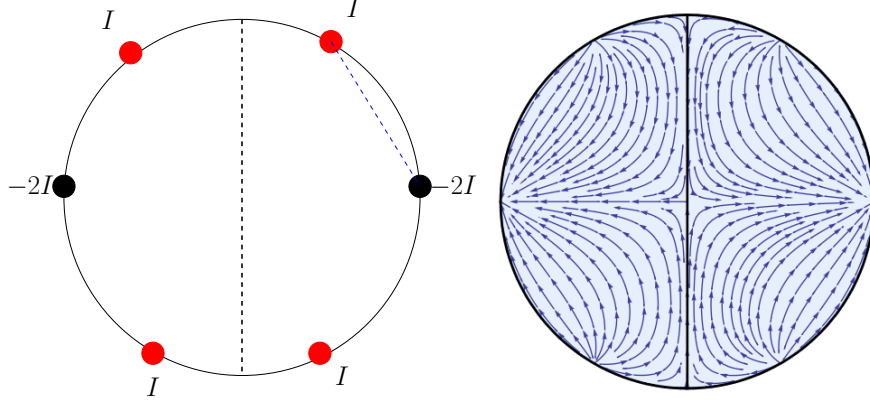
$$x = x_0 \exp \left(\sqrt{\frac{2\pi\Omega^2 R}{d \ln \left(\frac{16R^2}{9d^2} \right)}} t \right)$$

Here x_0 is the initial displacement of the water at $t = 0$. From this we recover the doubling time T_1 :

$$T_1 = (\ln 2) \sqrt{\frac{d \ln \left(\frac{16R^2}{9d^2} \right)}{2\pi\Omega^2 R}}$$

Case 2: With the Partition

The partition adds a new boundary condition because water must flow parallel to it. In addition, the setup is now only symmetric about a rotation by π . Since the semicircles are separated we also need to ensure volume conservation for each half. This motivates the solution on the left in the below figure, comprised of four sources I and two sinks $-2I$. The resulting streamlines are shown on the right.



Going clockwise from 1 o' clock, we sum each element's contribution the the potential difference across the blue dashed line:

$$\Delta V \approx \frac{\rho_{res} I}{2\pi d} \left(\ln \frac{R}{\frac{d}{2}} - 2 \ln \frac{\frac{d}{2}}{R} + \ln \frac{1}{\sqrt{3}} + \ln \frac{\sqrt{3}}{2} - 2 \ln \frac{2}{\sqrt{3}} + \ln \sqrt{3} \right)$$

$$\Delta V \approx \frac{\rho_{res} I}{2\pi d} \ln \frac{3\sqrt{3}R^3}{d^3}$$

The effective current here is $2I$ (only half the current lines are in the disk):

$$P = 2I\Delta V = \frac{\rho_{res} I^2}{\pi d} \ln \frac{3\sqrt{3}R^3}{d^3}$$

As before we map the dissipated power to the kinetic energy:

$$K = \frac{2\rho d^3}{\pi} \left(\ln \frac{3\sqrt{3}R^3}{d^3} \right) v^2$$

We get the potential energy by imagining four $d \times d \times \frac{x}{2}$ cuboids of water with centre of mass at $r = R + d - \frac{x}{4}$ disappearing from the "current source" holes, and two $d \times d \times x$ cuboids of water with centre of mass at $r = R + d + \frac{x}{2}$ appearing in the "current sink" holes. This gives us

$$U = 2 \left(-\frac{1}{2} (\rho x d^2) \Omega^2 \left(R + d + \frac{x}{2} \right)^2 \right) - 4 \left(-\frac{1}{2} \left(\frac{1}{2} \rho x d^2 \right) \Omega^2 \left(R + d - \frac{x}{4} \right)^2 \right)$$

$$U \approx \rho x \Omega^2 d^2 \left(-\frac{3}{2} x (R + d) \right) \approx -\frac{3}{2} \rho \Omega^2 R d^2 x^2$$

Invoking energy conservation, we have

$$\frac{2\rho d^3}{\pi} \left(\ln \frac{3\sqrt{3}R^3}{d^3} \right) \dot{x}^2 = \frac{3}{2} \rho \Omega^2 R d^2 x^2$$

$$\dot{x} = \sqrt{\frac{3\pi\Omega^2 R}{4d \ln\left(\frac{3\sqrt{3}R^3}{d^3}\right)}} x$$

$$x = x_0 \exp\left(\sqrt{\frac{3\pi\Omega^2 R}{4d \ln\left(\frac{3\sqrt{3}R^3}{d^3}\right)}} t\right)$$

This gives us

$$T_2 = (\ln 2) \sqrt{\frac{4d \ln\left(\frac{3\sqrt{3}R^3}{d^3}\right)}{3\pi\Omega^2 R}}$$

Finally, we obtain

$$T_2^2 - T_1^2 = \frac{(\ln 2)^2 d}{\pi\Omega^2 R} \left(\frac{4}{3} \ln\left(\frac{3\sqrt{3}R^3}{d^3}\right) - \frac{1}{2} \ln\left(\frac{16R^2}{9d^2}\right) \right)$$

$$T_2^2 - T_1^2 = \frac{(\ln 2)^2 d}{\pi\Omega^2 R} \left(\ln \frac{9R^4}{d^4} - \ln \frac{4R}{3d} \right)$$

$$T_2^2 - T_1^2 = \frac{(\ln 2)^2 d}{\pi\Omega^2 R} \ln \frac{27R^3}{4d^3}$$