

Physics Cup 2026 Problem 5

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1 Introduction

Let's start making qualitative considerations about the problem: of course, the ultraviolet light doesn't play any role apart from making the dye emit light. So we can make a sketch of the apparatus. We still know neither the value of the refraction index of the ball nor the position of the screen, so the sketch may not be realistic at all.

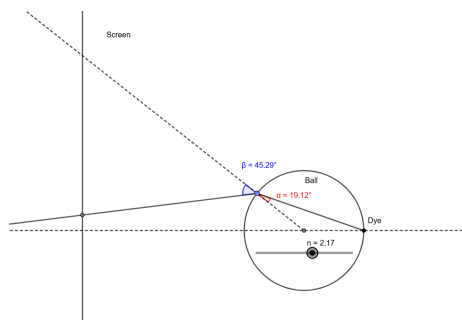


Figure 1: Sketch of apparatus.

The physics involved in this experiment is mainly Snell's law: a ray of light that passes through two materials with different refraction index follows the following law:

$$n_1 \sin \theta_1 = n_2 \sin \theta_2,$$

where θ_i is the angle formed by the light ray at the interface with the normal to the surface in the material of refraction index n_i . Hence we can write

$$\sin \beta = n \sin \alpha,$$

where α and β are the angles in the figures and n is the ratio between the refraction index of the crystal ball and air (or whatever is outside the ball).

The figure given by the problem shows the intensity that changes as the position on the screen changes. First of all, we can study the problem in two dimensions:

in fact there is cylindrical symmetry around the line that connects the center of the ball and the dye.

2 Non scattered rays

The problem says that the microscopic impurities inside the ball scatter light weakly, so it's reasonable to study the rays of light that aren't scattered.

Now it's about solving a geometrical problem: if the angle of emission of a ray is θ , what is the height $z(\theta)$ that the ray reaches?

First we notice that the angle of emission is the same angle that we find in the Snell's law (α in the figure), because the triangle formed by the ray and two radii is isosceles, so the two angles are equal.

The ray of light meets air at a height of

$$z_1 = R \sin(2\theta),$$

where R is the radius of the crystal ball. The other contribution to the total height is due to the distance of the screen. Let's call D the distance between the screen and the center of the ball. By simple trigonometry we get

$$z_2 = (D - R \cos(2\theta)) \tan(2\theta - \arcsin(n \sin \theta)).$$

So we have just found $z(\theta)$:

$$z = f(\theta) = R \sin(2\theta) + (D - R \cos(2\theta)) \tan(2\theta - \arcsin(n \sin \theta)).$$

The figure given by the problem qualitatively shows the function $I(z)$, that is the power by surface unity on the screen as a function of the height.

$$I(z) 2\pi z dz = \frac{dP}{d\Omega} d\Omega = I_0 d(2\pi R^2 (1 - \cos \theta)) = I_0 2\pi R^2 \sin \theta d\theta,$$

where $I_0 = \frac{dP}{d\Omega}$ is a constant.

$$\Rightarrow I(z) = I_0 \frac{R^2}{z} \sum_{\theta \in f^{-1}(z)} \frac{\sin \theta}{f'(\theta)}.$$

3 Image analysis

Let's try to get some useful information from the image provided. There are two interesting points to study: at some distance ell from the center, the points are very bright, and then they suddenly go darker; also, there is a maximum distance L where light gets.

I'll measure these distances with Paint: I'll measure all lengths in pixels.

The results are:

$$2R = d = 882 \pm 3,$$

$$2\ell = 258 \pm 1,$$

$$2L = 1222 \pm 1,$$

that is

$$\ell = (0.2925 \pm 0.0011)R,$$

$$L = (1.3855 \pm 0.0011)R.$$

Let's make a consideration: the bright points at $z = \ell$ might correspond to some θ where

$$f(\theta) = z,$$

$$f'(\theta) = 0,$$

so that the zero at the denominator makes $I(z)$ go to infinity.

Now, since the function $f(\theta)$ is pretty hard to study directly, I'll make use of Desmos to try to see the behavior of the function varying n and D .

What I want to do is to find the values of n and D that best fit our data, so we want a stationary point where the function equals ℓ and we want the function to be bounded into $[-L, L]$. I'll use "error bars" to help me judge the accuracy.

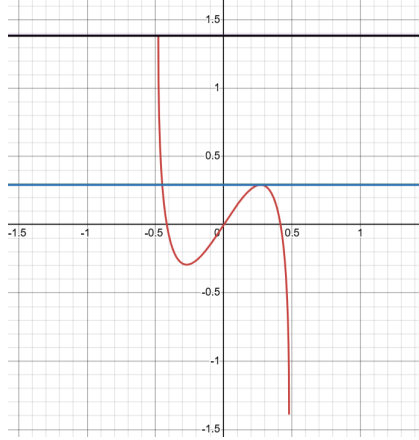


Figure 2: Plot of the function.

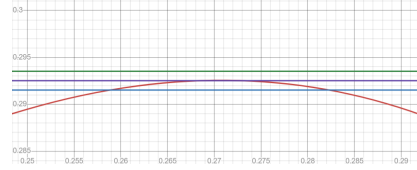


Figure 3: Zoom into the function.

After trying many values of n and D , I found the couple that best fits our data:

$$n = 2.1713,$$

$$D = 3.7041R.$$

However, there are other values of n for which there exists some D such that the function still respects the desired conditions. These values are in the interval $(2.1700, 2.1729)$, so we can give n an error:

$$n = 2.1713 \pm 0.0015.$$

4 Considerations

The fact that the light doesn't reach far points makes it clear that the scattering is weak; otherwise light could get everywhere. Also, if non scattered rays produce a divergence of intensity in $z = \ell$, and a fraction of these rays get scattered, there's still divergence. Now there's a problem: in $z = 0$, according to the formula for $I(z)$, there should be infinite intensity, because there are three values of θ such that $f(\theta) = 0$; although, in the figure it is pretty clear that at the center there is finite intensity. The most likely thing is the following: scattered rays follow random paths, so maybe the rays that hit the center are subject to interference with scattered rays, so there's no more divergence.

Now there are two types of light rays: those that, if not scattered, can exit the ball, and those can't; the latter are exactly those that can't make it the first time, in fact, if they are reflected, the angle formed with the surface the next time remains the same by simple geometry.

Hence, we can make an approximation: the first type of rays don't get scattered, while the second type necessarily gets scattered to exit the ball.

The solid angle that contains the escaping rays towards the screen is

$$\Omega = \int_0^{\theta_{\max}} 2\pi \sin x dx = 2\pi(1 - \cos \theta_{\max}),$$

where $\sin(n\theta_{\max}) = 1$. On the other side, the emission is similar: if the angle formed with the normal to the surface is less than $\theta_{\max}$ they go away, otherwise they get reflected and do the same as the others rays that get reflected until they scatter. So the solid angle of rays that escape from the other side is still Ω .

5 Final answer

Hence, the fraction of the light energy emitted by the fluorescent dye which departs the ball as scattered light is

$$\mu = \frac{4\pi - 2\Omega}{4\pi} = \cos \theta_{\max} = \sqrt{1 - \frac{1}{n^2}} \approx 0.8876.$$

An estimation of the error of μ is given by

$$\Delta\mu = \frac{d\mu}{dn} \Delta n = \frac{\Delta n}{n^3 \sqrt{1 - n^{-2}}} \approx 2 \cdot 10^{-4} \ll 10^{-2}.$$

Hence, we can write the result in the following form:

$$\boxed{\frac{E_{\text{scattered}}}{E_{\text{total}}} \approx 0.89}$$