

Physics Cup 2026: Problem 3

Shaocong Sun
18202535545@163.com

January 2026

1 Circuit equations

Based on the symmetry shown in the circuit structure, we can simplify the infinite circuit to the structure depicted in Figure 1. Here, R_x and R_y are used to represent the complex impedances of capacitor C and inductor L . The current and voltage marked in the figure satisfy Equations (1) and (2).

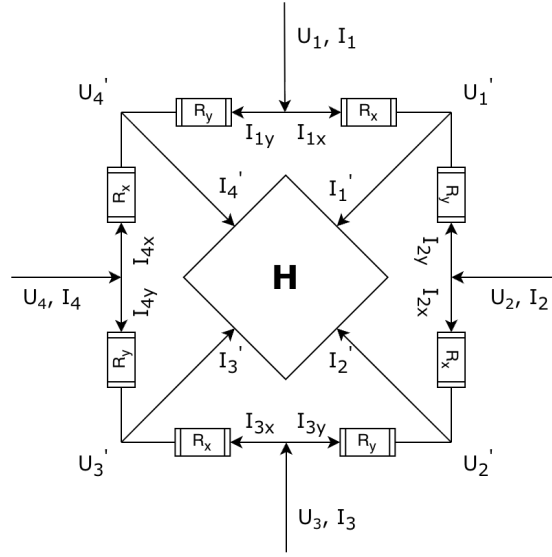


Figure 1: Circuit

$$\begin{aligned}
 I_1 &= I_{1x} + I_{1y}, & I'_1 &= I_{1x} + I_{2y}; \\
 I_2 &= I_{2x} + I_{2y}, & I'_2 &= I_{2x} + I_{3y}; \\
 I_3 &= I_{3x} + I_{3y}, & I'_3 &= I_{3x} + I_{4y}; \\
 I_4 &= I_{4x} + I_{4y}, & I'_4 &= I_{4x} + I_{1y};
 \end{aligned} \tag{1}$$

$$\begin{aligned}
U_1 &= U'_1 + R_x \cdot I_{1x} = U'_4 + R_y \cdot I_{1y}; \\
U_2 &= U'_2 + R_x \cdot I_{2x} = U'_1 + R_y \cdot I_{2y}; \\
U_3 &= U'_3 + R_x \cdot I_{3x} = U'_2 + R_y \cdot I_{3y}; \\
U_4 &= U'_4 + R_x \cdot I_{4x} = U'_3 + R_y \cdot I_{4y};
\end{aligned} \tag{2}$$

We can define

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad J^{-1} = J^3 = J^t, \quad J^4 = I_d$$

$$\vec{U} = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix}, \quad \vec{I} = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix}, \quad \vec{U}' = \begin{pmatrix} U'_1 \\ U'_2 \\ U'_3 \\ U'_4 \end{pmatrix}, \quad \vec{I}' = \begin{pmatrix} I'_1 \\ I'_2 \\ I'_3 \\ I'_4 \end{pmatrix}, \quad \vec{I}_x = \begin{pmatrix} I_{1x} \\ I_{2x} \\ I_{3x} \\ I_{4x} \end{pmatrix}, \quad \vec{I}_y = \begin{pmatrix} I_{1y} \\ I_{2y} \\ I_{3y} \\ I_{4y} \end{pmatrix}$$

Then, Equations (1) and (2) can be rewritten as Equations (3) and (4). Suppose that the current-voltage relationship of this infinite circuit can be described by the matrix H , that is, Equation (5).

$$\vec{I} = \vec{I}_x + \vec{I}_y, \quad \vec{I}' = \vec{I}_x + J^{-1} \cdot \vec{I}_y \tag{3}$$

$$\vec{U} = \vec{U}' + R_x \cdot \vec{I}_x = J \cdot \vec{U}' + R_y \cdot \vec{I}_y \tag{4}$$

$$\vec{I} = H \cdot \vec{U}, \quad \vec{I}' = H \cdot \vec{U}' \tag{5}$$

2 Solving for H

Now let's solve for H . From the symmetry of the circuit structure, all matrices involved in the following derivation (including H) can be expressed as polynomials in J . Therefore, these matrices commute with each other. Using I' and U' to express $\vec{I}_x$, we obtain Equation (6). Similarly, using I and U to express $\vec{I}_x$, we obtain Equation (7).

$$\vec{I}_x = (R_x I_d + R_y J)^{-1} [(J - I_d) \vec{U}' + R_y J \vec{I}'] \tag{6}$$

$$\vec{I}_x = (R_y I_d + R_x J)^{-1} [(J - I_d) \vec{U} + R_y \vec{I}] \tag{7}$$

I_x expressed in two forms must be equal. So we get Equation (8).

$$(R_x I_d + R_y J)^{-1} \cdot [(J - I_d) \vec{U}' + R_y J \vec{I}'] = (R_y I_d + R_x J)^{-1} \cdot [(J - I_d) \vec{U} + R_y \vec{I}] \tag{8}$$

Combined with $\vec{I} = H \vec{U}$ and $\vec{I}' = H \vec{U}'$, we obtain Equation (9).

$$(R_y I_d + R_x J) \cdot (J - I_d + R_y J H) \cdot \vec{U}' = (R_x I_d + R_y J) \cdot (J - I_d + R_y H) \cdot \vec{U} \tag{9}$$

Substituting the relationship between $\vec{U}$ and $\vec{U}'$ in Equation (10) into Equation (9), and after a series of calculations, we obtain the final equation for H , Equation (11).

$$\begin{aligned}
\vec{U} &= \vec{U}' + R_x \cdot \vec{I}_x \\
&= \vec{U}' + R_x \cdot (R_x I_d + R_y J)^{-1} \cdot [(J - I_d)\vec{U}' + R_y J \vec{I}'] \\
&= [I_d + R_x \cdot (R_x I_d + R_y J)^{-1} \cdot (J - I_d + R_y J H)] \cdot \vec{U}'
\end{aligned} \tag{10}$$

$$R_x R_y H^2 = -J^{-1} \cdot (J - I_d)^2 \tag{11}$$

$$R_x R_y H^2 = 2I_d - J^{-1} - J = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} \tag{12}$$

Let $A = 2I_d - J - J^3$. Since A is a real symmetric matrix, H is obtained through eigenvalue decomposition as shown in Equation (13).

$$\det(\lambda I_d - A) = 0 \quad \Rightarrow \quad \lambda = 0, 2, 2, 4$$

$$\lambda = 0: \quad \vec{u}_1 = \frac{1}{2}(1, 1, 1, 1)^t$$

$$\lambda = 2: \quad \vec{u}_2 = \frac{1}{\sqrt{2}}(-1, 0, 1, 0)^t, \quad \vec{u}_3 = \frac{1}{\sqrt{2}}(0, -1, 0, 1)^t$$

$$\lambda = 4: \quad \vec{u}_4 = \frac{1}{2}(1, -1, 1, -1)^t$$

$$\sqrt{R_x R_y} H = \sqrt{2} \vec{u}_2 \vec{u}_2^t + \sqrt{2} \vec{u}_3 \vec{u}_3^t + 2 \vec{u}_4 \vec{u}_4^t = \frac{1}{2} \begin{pmatrix} \sqrt{2}+1 & -1 & 1-\sqrt{2} & -1 \\ -1 & \sqrt{2}+1 & -1 & 1-\sqrt{2} \\ 1-\sqrt{2} & -1 & \sqrt{2}+1 & -1 \\ -1 & 1-\sqrt{2} & -1 & \sqrt{2}+1 \end{pmatrix} \tag{13}$$

3 Final result

At this point, we have obtained the current-voltage relationship matrix H that can describe the current and voltage of any infinite circuit that conforms to the structure shown in Figure 1. H can be adapted to any impedance value of R_x and R_y . Finally, we substitute the scenario of this problem to obtain the final result.

For any frequency ω , the complex impedances of the inductor and capacitor are as shown in Equation (14). It is noted that after substituting the actual values of R_x and R_y , the resulting matrix H is independent of ω . This indicates that the performance of the circuit to the outside can, to some extent, be equivalent to that of a pure resistive circuit.

$$R_x = \frac{1}{j\omega C}, \quad R_y = j\omega L, \quad \sqrt{R_x R_y} = \sqrt{\frac{L}{C}} \tag{14}$$

The input boundary conditions are shown in Equation (15). Using $\vec{I} = H\vec{U}$ to get the final result of $I(t)$ in Equation (18).

$$U_3 = 0, \quad U_1 = V_0, \quad I_2 = I_4 = 0, \quad I_1 = -I_3 = I(t) \quad (15)$$

$$\begin{pmatrix} I(t) \\ 0 \\ -I(t) \\ 0 \end{pmatrix} = H \begin{pmatrix} V_0 \\ U_2 \\ 0 \\ U_4 \end{pmatrix} = \frac{1}{2} \sqrt{\frac{C}{L}} \begin{pmatrix} \sqrt{2}+1 & -1 & 1-\sqrt{2} & -1 \\ -1 & \sqrt{2}+1 & -1 & 1-\sqrt{2} \\ 1-\sqrt{2} & -1 & \sqrt{2}+1 & -1 \\ -1 & 1-\sqrt{2} & -1 & \sqrt{2}+1 \end{pmatrix} \begin{pmatrix} V_0 \\ U_2 \\ 0 \\ U_4 \end{pmatrix} \quad (16)$$

$$\begin{cases} -V_0 + (\sqrt{2}+1)U_2 + (1-\sqrt{2})U_4 = 0 \\ -V_0 + (1-\sqrt{2})U_2 + (\sqrt{2}+1)U_4 = 0 \end{cases} \Rightarrow U_2 = U_4 = \frac{V_0}{2} \quad (17)$$

$$I(t) = \frac{1}{2} \sqrt{\frac{C}{L}} [(\sqrt{2}+1)V_0 - U_2 - U_4] = \sqrt{\frac{C}{2L}} V_0 \quad (18)$$