Physics Cup Problem 1

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Note that $r \ll R, L$, so we ignore the effect of charges on the rod throughout this problem. The system goes through the following processes:

1. Due to the Lorentz forces acting on charges in the metal, the dumbbell will be polarized in the direction of $\mathbf{v} \times \mathbf{B}$. This will create a homogeneous electric field within the dumbbell to ensure that the force acting on free charges in the dumbbell $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \mathbf{0}$ (Figure 1).

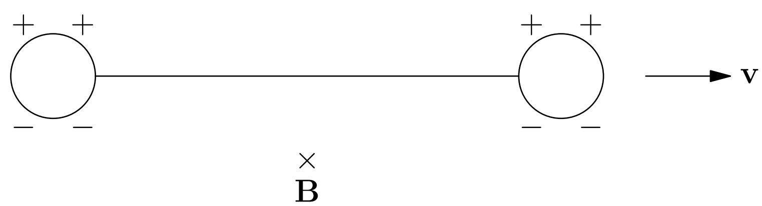

Figure 1: Initial polarization of the dumbbell

2. The position of the dumbbell is unstable. Slightly tilting the dumbbell will cause one side of the dumbbell to gain a positive charge and the other side to gain a negative charge. The dumbbell then experiences a torque that moves it away from equilibrium. Afterwards, the dumbbell will start to oscillate and lose energy through emitting electromagnetic radiation (Figure 2).

3. As electromagnetic radiation carries the oscillation energy away from the dumbbell, the amplitude will decrease to 0. At that point, the dumbbell will be in a stable equilibrium moving at velocity $\mathbf{u}$ (Figure 3).

Note that throughout the oscillation, although energy/momentum is not conserved, the following quantity is conserved:

$$\sum_q q\mathbf{r} \times \mathbf{B} - \mathbf{p}, \tag{*}$$

where q is a charge, $\mathbf{r}$ is its position vector relative to the center $\mathbf{r}_C$ of the dumbbell, and $\mathbf{p}$ is the total kinetic momentum of the dumbbell.

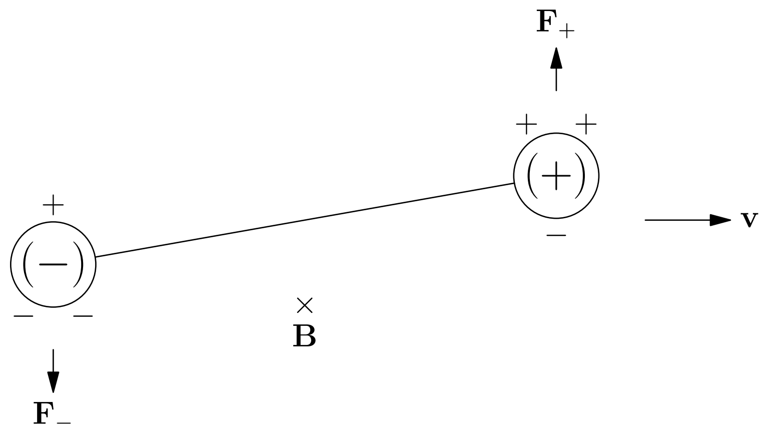

Figure 2: A tilted dumbbell will experience a torque

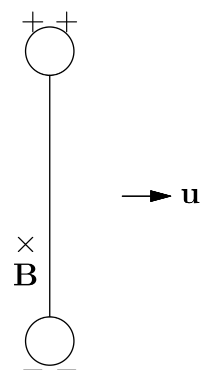

Figure 3: New stable equilibrium after oscillation amplitude decreases to 0.

Proof. For a single point charge q, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(q(\mathbf{r}+\mathbf{r}_C)\times\mathbf{B}-\mathbf{p}_q\right)=q\mathbf{v}\times\mathbf{B}-\mathbf{F}_{\text{net},q}=-\mathbf{F}_{\text{internal},q}.$$

Summing over all charges (i.e., atomic nuclei and electrons) in the dumbbell, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_q q(\mathbf{r}+\mathbf{r}_C)\times\mathbf{B}-\mathbf{p}\right)=-\sum_q \mathbf{F}_{\text{internal},q}=\mathbf{0}.$$

But $\sum_q q\mathbf{r}_C\times\mathbf{B}=\mathbf{0}$ since the net charge of the dumbbell $\sum_q q=0$. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_q q\mathbf{r}\times\mathbf{B}-\mathbf{p}\right)=\mathbf{0}.$$

□

We therefore solve the problem in five steps:

1. Find the initial charge distribution in terms of $\mathbf{B}, \mathbf{v}, L, R$.

2. Find the final orientation of the dumbbell.
3. Find the final charge distribution in terms of $\mathbf{B}, \mathbf{u}, L, R$.
4. Find the change in $\sum_q q\mathbf{r} \times \mathbf{B}$ from the initial charge distribution to the final charge distribution in terms of $\mathbf{B}, \mathbf{v}, \mathbf{u}, L, R$.
5. Setting $\Delta \mathbf{p} = m(\mathbf{u} - \mathbf{v})$ equal to $\Delta \left(\sum_q q\mathbf{r} \times \mathbf{B} \right)$ found in step 4, we solve for $\mathbf{u}$ in terms of $\mathbf{B}, \mathbf{v}, L, R$.

1 Initial charge distribution

Consider a single copper shell with radius R moving at velocity $\mathbf{v}$. The effect of $\mathbf{B}$ on the charges in the ball is equivalent to that of an electric field $\mathbf{E}_{\text{eff}} = \mathbf{v} \times \mathbf{B}$. The effective electric field $\mathbf{E}_{\text{eff}}$ polarizes the ball so that it now has surface charge density distribution $\sigma(\theta)$ (Figure 4).

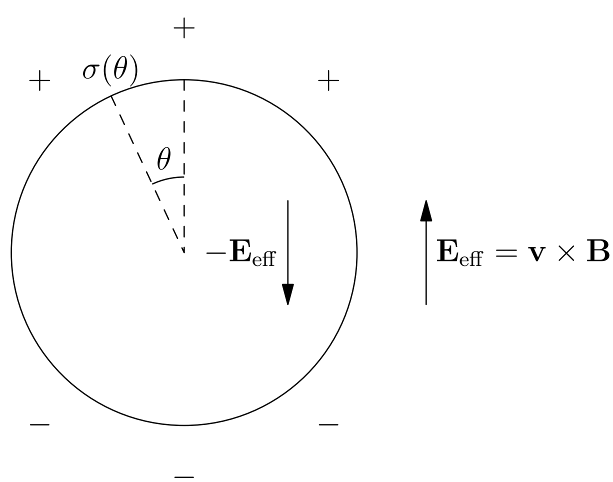

Figure 4: Polarized copper ball

Approximate the spherical copper shell as the superposition of two balls of radius R with uniform charge density $\pm\rho$. If we denote by $\mathbf{d}$ ($d = |\mathbf{d}| \ll R$) the vector from the center of the negatively-charged ball to the center of the positively-charged ball, then the region inside the shell has the uniform electric field

$$\mathbf{E} = \frac{\rho V}{4\pi\epsilon_0 R^3}(\mathbf{r} - \mathbf{r}_+) - \frac{\rho V}{4\pi\epsilon_0 R^3}(\mathbf{r} - \mathbf{r}_-) = \frac{\rho V}{4\pi\epsilon_0 R^3}(\mathbf{r}_- - \mathbf{r}_+) = -\frac{\rho}{3\epsilon_0}\mathbf{d},$$

where we have used $\mathbf{d} = \mathbf{r}_- - \mathbf{r}_+$ and $V = \frac{4}{3}\pi R^3$. The interior of the shell will have $+\rho$ and $-\rho$ canceling out, whereas on the surface there will be a charge distribution $\sigma(\theta) = \mathbf{P} \cdot \hat{\mathbf{r}}(\theta) = \rho d \cos\theta$, where $\mathbf{P}$ is the polarization density.

From

$$\mathbf{E}_{\text{eff}} = -\mathbf{E} = \frac{\rho}{3\epsilon_0}\mathbf{d},$$

we get

$$\rho d = 3\epsilon_0 E_{\text{eff}} = 3\epsilon_0 vB.$$

Therefore,

$$\sigma(\theta) = \rho d \cos\theta = 3\epsilon_0 vB \cos\theta.$$

The electric field generated by the charge on the shell is equivalent to an electric dipole moment

$$\begin{aligned}\mathbf{p}_{\text{d}} &= \rho V \mathbf{d} \\ p_{\text{d}} &= 3\epsilon_0 vBV = 4\pi\epsilon_0 vBR^3.\end{aligned}$$

Now, return to the dumbbell, which has two such shells. Here, the electric fields due to the dipole moments of the two balls will influence each other and alter their surface charge distributions, and solving for the resultant electric field using the method of image charges would require infinitely many image dipoles. However, we realize that the image dipole of the dipole $\mathbf{p}_{\text{d}}$ of one spherical shell with respect to the other shell is

$$\mathbf{p}'_{\text{d}} = q'\mathbf{d}' = \left(-\frac{R}{L+2R}q\right)\left(\frac{R^2}{(L+2R)^2}\mathbf{d}\right) = -\left(\frac{R}{L+2R}\right)^3 \mathbf{d} \approx -0.0006\mathbf{p}_{\text{d}}.$$

Since $p'_{\text{d}} \ll p_{\text{d}}$, we may ignore the image dipoles. Equivalently, we may ignore the effect of the interaction between the spheres on the charge distributions on the spheres.

The electric fields from the two copper shells would also affect the charge distribution on the copper rod connecting them. However, the induced charge (on the upper half of the rod) is

$$Q_{\text{rod}} \sim \epsilon_0 E_{\text{rod}} A_{\text{rod}} \approx \epsilon_0 \left(\frac{R}{L/2}\right)^3 vB \cdot r^2 L = \epsilon_0 vBR^3 \frac{8r^2}{L^2} \ll \epsilon_0 vBR^3 \sim Q_{\text{shell}}.$$

So we may ignore the induced charge on the rod.

In conclusion, the initial charge distribution on each sphere is

$$\sigma_{\text{i}}(\theta) = 3\epsilon_0 vB\cos\theta. \tag{1}$$

2 Final orientation of the dumbbell

At stable equilibrium, the rod will be perpendicular to $\mathbf{u}$—otherwise, the rod will experience a torque as in Figure 2. Since the effective electric field $\mathbf{u} \times \mathbf{B}$ is parallel to the rod, the induced charge distribution will be axially symmetric. Therefore, in calculating $\sum_q q\mathbf{r} \times \mathbf{B}$, the component parallel to the rod (arising from the transverse component of $\mathbf{r}$) cancels out. This leaves only $\sum_q q\mathbf{r}_\parallel \times \mathbf{B}$, which is parallel to $\mathbf{u}$. Therefore, $\sum_q q\mathbf{r} \times \mathbf{B} \parallel \mathbf{u}$, meaning $\left(\sum_q q\mathbf{r} \times \mathbf{B} - \mathbf{p}\right)_{\text{f}}$ is parallel to $\mathbf{p}_{\text{f}}$.

Consider the initial configuration where the rod is parallel to $\mathbf{v}$. The charge distribution is symmetric about the plane perpendicular to the rod through its midpoint. Therefore, in calculating $\sum_q q\mathbf{r} \times \mathbf{B}$, the component parallel to $\mathbf{v} \times \mathbf{B}$ (arising from the component of $\mathbf{r}$ parallel to the rod) cancels out. This leaves only $\sum_q q\mathbf{r}_\perp \times \mathbf{B}$, which is parallel to $\mathbf{v}$. Therefore, $\sum_q q\mathbf{r} \times \mathbf{B} \parallel \mathbf{v}$, meaning $\left(\sum_q q\mathbf{r} \times \mathbf{B} - \mathbf{p}\right)_\mathrm{i}$ is parallel to $\mathbf{p}_\mathrm{i}$.

Since $\left(\sum_q q\mathbf{r} \times \mathbf{B} - \mathbf{p}\right)_\mathrm{f} = \left(\sum_q q\mathbf{r} \times \mathbf{B} - \mathbf{p}\right)_\mathrm{i} \neq \mathbf{0}$ (it points opposite to $\mathbf{v}$ or $\mathbf{u}$), we then have $\mathbf{p}_\mathrm{f} \parallel \mathbf{p}_\mathrm{i}$. In other words, $\mathbf{u} \parallel \mathbf{v}$.

3 Final charge distribution

First, ignore the induced charge on the copper rod. We will later come back and justify this assumption.

Consider the system with just the two copper shells. They are each polarized into a dipole $\mathbf{p}_\mathrm{d}$ in the effective electric field, and as we've shown before in Section 1, the dipole of each shell barely influences the other shell.

In addition, the top shell will gain a net positive charge relative to the bottom shell so that they're at the same potential (considering the effective electric field to contribute to potential as well). The additional induced charge can be found using infinitely many image charges (Figure 5): the image charges of $1'$, $2'$,... relative to the top shell are 2, 3,..., and the image charges of 1, 2,... relative to the bottom shell are $2'$, $3'$,...

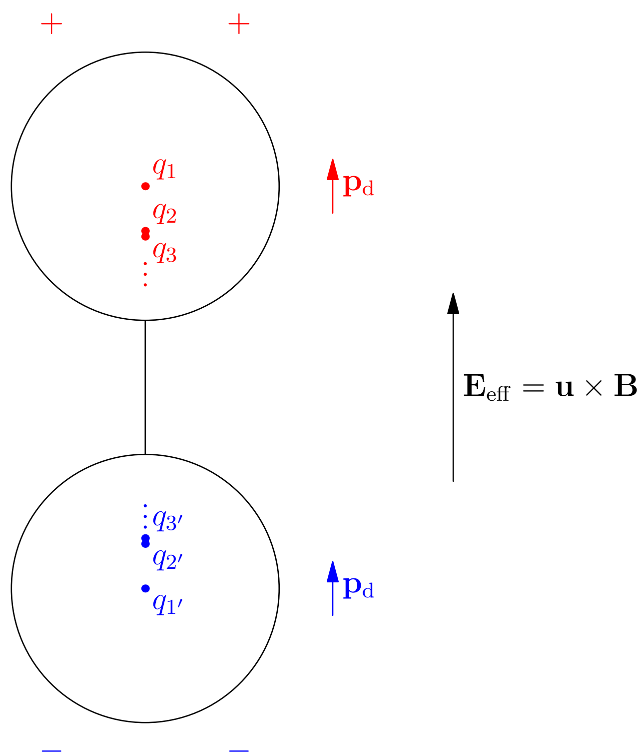

Figure 5: Final charge distribution equivalent to two dipoles and many image charges

Consider, for example, the additional surface charge distribution on the top ball due to

$1'$:

$$\sigma_{1'\mathrm{T}}(\theta) = \epsilon_0 \left|\mathbf{E}_{1'} + \mathbf{E}_2\right| = \frac{-q_{1'}}{4\pi R^2} \frac{x(1-x^2)}{(1+x^2+2x\cos\theta)^{\frac{3}{2}}},$$

where $x = R/(L+2R) = 1/12$ (see the Appendix (Section 6) for the derivation). Eventually, we'll be computing $\sum_q q\mathbf{r} \times \mathbf{B} = \sum_q q\mathbf{r}_\parallel \times \mathbf{B}$. Hopefully, the distribution $\sigma_{1'\mathrm{T}}$ is close enough to a uniform distribution $\frac{q_{1'}x}{4\pi R^2}$ that we can assume it is. Indeed, the difference in $\sum_q q\mathbf{r}_\parallel \times \mathbf{B}$ between the $\sigma_{1'\mathrm{T}}(\theta)$ distribution and the uniform distribution is

$$\begin{aligned}
\iint_{\text{shell}} \left(\sigma_{1'\mathrm{T}}(\theta) - \frac{-q_{1'}x}{4\pi R^2}\right)(R\cos\theta)\,\mathrm{d}S &= \frac{q_{1'}x}{4\pi R} \iint_{\text{shell}} \left(\frac{1-x^2}{(1+x^2+2x\cos\theta)^{\frac{3}{2}}} - 1\right)\cos\theta\,\mathrm{d}S \\
&= \frac{-q_{1'}xR}{2} \int_0^\pi \left(\frac{1-x^2}{(1+x^2+2x\cos\theta)^{\frac{3}{2}}} - 1\right)\sin\theta\cos\theta\,\mathrm{d}\theta \\
&\approx 0.0069 q_{1'}R = -0.0069 q_1 R.
\end{aligned}$$

This is negligible compared to the $q_1(L/2+R)$ contributed by the uniform charge distribution on the top shell corresponding to charge 1. Therefore, it is valid to approximate $\sigma_{1'\mathrm{T}}(\theta)$ as simply a uniform distribution. Similarly, the induced charges on the top sphere by the other charges n' can be approximated as uniform:

$$\sigma_{n'\mathrm{T}}(\theta) \approx \frac{-q_{n'}x}{4\pi R^2} = \frac{q_n x}{4\pi R^2}$$

By symmetry, the induced charges on the bottom sphere by the charges n can also be approximated as uniform:

$$\sigma_{n\mathrm{B}}(\theta) \approx -\frac{q_n x}{4\pi R^2}.$$

Now, we need to find q_n. To do this, we use the fact that the two shells are at the same potential. Let the potential at the center of the dumbbell be zero—then the potential of both shells are zero. The potential at the bottom of the top shell consists of four terms—the potential due to the effective electric field, the potential due to the electric dipole of the top sphere, the potential due to the electric dipole of the bottom sphere, and the potential due to q_1. (Note that the potentials due to q_2 and $q_{1'}$, due to q_3 and $q_{2'}$, etc. all cancel out. This is because q_{n+1} is the image charge of $q_{n'}$; a charge and its image charge result in zero potential at the surface of the conductor in question.) Therefore,

$$\begin{aligned}
0 &= U_\mathrm{T} \\
0 &= -\frac{1}{2}E_\mathrm{eff}L - \frac{p_\mathrm{d}}{4\pi\epsilon_0 R^2} + \frac{p_\mathrm{d}}{4\pi\epsilon_0(R+L)^2} + \frac{q_1}{4\pi\epsilon_0 R} \\
0 &= -\frac{1}{2}uBL - uBR + \frac{uBR^3}{(R+L)^2} + \frac{q_1}{4\pi\epsilon_0 R} \qquad (p_\mathrm{d} = 4\pi\epsilon_0 uBR^3) \\
0 &= -uBR\left[\frac{L}{2R} + 1 - \left(\frac{R}{R+L}\right)^2\right] + \frac{q_1}{4\pi\epsilon_0 R} \\
q_1 &= 4\pi\epsilon_0 uBR^2\left[\frac{L}{2R} + 1 - \left(\frac{R}{R+L}\right)^2\right]
\end{aligned}$$

Given q_1, we can now find q_n. Let $y_n R$ be the distance between charge n and the center of the top shell ($y_1 = 0$). Then $y_n R$ is also the distance between charge n' and the center of the bottom shell. Since q_{n+1} is the image charge of $q_{n'}$ with respect to the top shell, we have

$$y_{n+1} = \frac{1}{x^{-1} - y_n}.$$

Then y_n is an increasing sequence with limit point y_∞ satisfying

$$y_\infty = \frac{1}{x^{-1} - y_\infty},$$

solving which gives

$$y_\infty = \frac{1 - \sqrt{1 - 4x^2}}{2x}.$$

Therefore, each y_n ($n \geq 2$) satisfies

$$\begin{aligned} y_2 &\leq y_n < y_\infty \\ x &\leq y_n < y\frac{1 - \sqrt{1 - 4x^2}}{2x}. \end{aligned}$$

Now, note that q_n satisfies the recurrence

$$q_{n+1} = -y_{n+1} q_{n'} = y_{n+1} q_n,$$

so

$$\begin{aligned} q_n &= q_1 \prod_{k=2}^{n} y_k \\ q_1 x^{n-1} \leq q_n &< q_1 \left(\frac{1 - \sqrt{1 - 4x^2}}{2x} \right)^{n-1}. \end{aligned}$$

Using the ranges for q_n, we can approximate the total charge $Q = \sum_{n=1}^{\infty} q_n$ on the top sphere and hence find the surface charge distribution due to Q:

$$\begin{aligned} q_1 \sum_{n=1}^{\infty} x^{n-1} \leq Q &< q_1 \sum_{n=1}^{\infty} \left(\frac{1 - \sqrt{1 - 4x^2}}{2x} \right)^{n-1} \\ \frac{q_1}{1 - x} \leq Q &< \frac{q_1}{1 - \frac{1 - \sqrt{1 - 4x^2}}{2x}}. \end{aligned}$$

Note that $1 - x \approx 0.91667$ and $1 - \frac{1 - \sqrt{1 - 4x^2}}{2x} \approx 0.91608$ differ by less than 0.001, so we can approximate Q as either endpoint of the range given above. Before, we have overestimated $\sum_q q\mathbf{r}_\parallel \times \mathbf{B}$ by approximating $\sigma_{n'\mathrm{T}}(\theta)$ as a uniform distribution, so now we should slightly underestimate Q to cancel out some of the overestimation:

$$Q \approx \frac{q_1}{1 - x}.$$

Now, we can find the corresponding surface charge density distribution due to Q. This distribution incorporates not only q_1 but also the induced charge distributions $\sigma_{n'\mathrm{T}}(\theta)$ $(n \geq 1)$, which are approximately uniform distribution. The distribution of Q is therefore approximately uniform:

$$\begin{aligned}
\sigma_Q(\theta) &\approx \frac{Q}{4\pi R^2} \\
&\approx \frac{q_1}{4\pi R^2(1-x)} \\
&= 4\pi\epsilon_0 uBR^2 \left[\frac{L}{2R} + 1 - \left(\frac{R}{R+L}\right)^2\right] / [4\pi R^2(1-x)] \\
&= \epsilon_0 uB \frac{\frac{L}{2R} + 1 - \left(\frac{R}{R+L}\right)^2}{1 - \frac{R}{2R+L}} \\
&\approx \epsilon_0 uB \frac{\frac{L}{2R} + 1}{1 - \frac{R}{2R+L}} = \epsilon_0 uB \frac{(2R+L)^2}{2R(R+L)}.
\end{aligned}$$

(We have made the approximation $\left(\frac{R}{R+L}\right)^2 \ll 1$, which results in a slight overestimate; this can slightly compensate for the underestimation of $\sum_q q\mathbf{r}_\parallel \times \mathbf{B}$ resulting from neglecting induced charges on the rod (see the end of this section).) Now, we add to $\sigma_Q(\theta)$ the charge distribution creating the dipole moment (i.e., (1) but with u instead of v) to get the net distribution of charge on the top sphere:

$$\sigma_{\mathrm{f,T}}(\theta) = \epsilon_0 uB \frac{(2R+L)^2}{2R(R+L)} + 3\epsilon_0 uB \cos\theta. \tag{2}$$

The charge distribution on the bottom sphere is similar, consisting of $-\sigma_Q(\theta)$ by symmetry and also the dipole distribution (1):

$$\sigma_{\mathrm{f,B}}(\theta) = -\epsilon_0 uB \frac{(2R+L)^2}{2R(R+L)} + 3\epsilon_0 uB \cos\theta. \tag{3}$$

It remains to validate our assumption that the induced charge on the copper rod connecting the two spheres can be neglected. Note that if there is no charge in the rod, then the potential is zero at the top sphere and at the middle of the rod, but is negative in the middle of the upper half of the rod. So there will be positive induced charge—assume it is approximately evenly distributed with line density λ from $L/8$ to $3L/8$ (Figure 6).

Then the potential at $L/4$ due to the line distribution of charge is

$$\begin{aligned}
U_{L/4} &= \int_{-\frac{L}{8}}^{\frac{L}{8}} \frac{\lambda\,\mathrm{d}y}{4\pi\epsilon_0\sqrt{r^2+y^2}} \\
&= \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{L}{8r} + \sqrt{\left(\frac{L}{8r}\right)^2 + 1}\right)
\end{aligned}$$

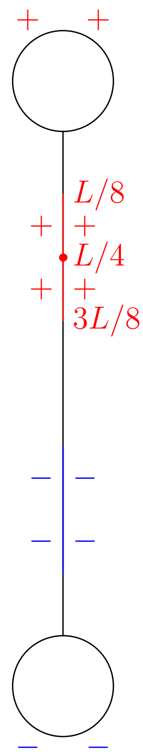

Figure 6: There is a small amount of induced charge on the rod.

This should cancel out with the potential due to $\mathbf{E}_{\text{eff}}$ and the charges on the spheres, which must be larger than $-LE_{\text{eff}}/4$. As a reasonable approximation, we take

$$\begin{aligned} U_{L/4} &\approx \frac{1}{8} L E_{\text{eff}} \\ \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{L}{8r} + \sqrt{\left(\frac{L}{8r}\right)^2 + 1}\right) &\approx \frac{1}{8} L E_{\text{eff}} \\ \lambda &\approx \frac{\pi\epsilon_0 E_{\text{eff}} L}{4\ln\left(\frac{L}{8r} + \sqrt{\left(\frac{L}{8r}\right)^2+1}\right)} \\ q = \frac{L}{4}\lambda &\approx \frac{\pi\epsilon_0 E_{\text{eff}} L^2}{16\ln\left(\frac{L}{8r} + \sqrt{\left(\frac{L}{8r}\right)^2+1}\right)}. \end{aligned}$$

Note that

$$Q \gtrsim q_1 = 4\pi\epsilon_0 E_{\text{eff}} R^2 \left[\frac{L}{2R} + 1 - \left(\frac{R}{R+L}\right)^2\right] \gtrsim 2\pi\epsilon_0 E_{\text{eff}} L R,$$

so

$$\frac{q}{Q} \lesssim \frac{L}{32R\ln\left(\frac{L}{8r} + \sqrt{\left(\frac{L}{8r}\right)^2+1}\right)} \approx 0.057 \ll 1.$$

Therefore, it is reasonable to neglect the induced charge q on the upper half of the rod. It is similarly reasonable to neglect the induced charge $-q$ on the bottom half of the rod.

4 Determine the change in $\sum_q q\mathbf{r} \times \mathbf{B}$

For the initial configuration,

$$\begin{aligned}
\left(\sum_q q\mathbf{r} \times \mathbf{B}\right)_{\mathrm{i}} &= \left(\sum_q q\mathbf{r}_\perp \times \mathbf{B}\right)_{\mathrm{i}} \\
&= 2 \iint_{\text{sphere}} \sigma_{\mathrm{i}}(\theta)(R\cos\theta)\,\mathrm{d}S\,(\hat{\mathbf{y}} \times \mathbf{B}) \\
&= 2 \iint_{\text{sphere}} 3\epsilon_0 vBR\cos^2\theta\,\mathrm{d}S\,(-B\hat{\mathbf{x}}) \qquad \text{(from (1))} \\
&= -2\epsilon_0 vB^2R \iint_{\text{sphere}} 3\cos^2\theta\,\mathrm{d}S\,\hat{\mathbf{x}} \\
&= -8\pi\epsilon_0 vB^2R^3\hat{\mathbf{x}},
\end{aligned}$$

where we have used

$$\begin{aligned}
\iint_{\text{sphere}} 3\cos^2\theta\,\mathrm{d}S &= \int_0^\pi 3\cos^2\theta \cdot 2\pi R^2 \sin\theta\,\mathrm{d}\theta \\
&= 2\pi R^2 \left[-\cos^3\theta\right]_0^\pi \\
&= 4\pi R^2.
\end{aligned}$$

For the final configuration,

$$\begin{aligned}
\left(\sum_q q\mathbf{r} \times \mathbf{B}\right)_{\mathrm{f}} &= \left(\sum_q q\mathbf{r}_\parallel \times \mathbf{B}\right)_{\mathrm{f}} \\
&= \left[\iint_{\text{sphere}} \sigma_{\mathrm{f,T}}(\theta)\left(\frac{L}{2} + R\cos\theta\right)\mathrm{d}S + \iint_{\text{sphere}} \sigma_{\mathrm{f,B}}(\theta)\left(-\frac{L}{2} + R\cos\theta\right)\mathrm{d}S\right](\hat{\mathbf{y}} \times \mathbf{B}) \\
&= \left[\iint_{\text{sphere}} \left(\epsilon_0 uB\frac{(2R+L)^2}{2R(R+L)} + 3\epsilon_0 uB\cos\theta\right)\left(\frac{L}{2} + R\cos\theta\right)\mathrm{d}S\right. \\
&\quad \left. + \iint_{\text{sphere}} \left(-\epsilon_0 uB\frac{(2R+L)^2}{2R(R+L)} + 3\epsilon_0 uB\cos\theta\right)\left(-\frac{L}{2} + R\cos\theta\right)\mathrm{d}S\right](-B\hat{\mathbf{x}}) \\
&= \left[\iint_{\text{sphere}} \left(\epsilon_0 uB\frac{(2R+L)^2}{2R(R+L)}\frac{L}{2} + 3\epsilon_0 uB\cos\theta \cdot R\cos\theta\right)\mathrm{d}S\right. \\
&\quad + \iint_{\text{sphere}} \left(\epsilon_0 uB\frac{(2R+L)^2}{2R(R+L)}R\cos\theta + 3\epsilon_0 uB\cos\theta \cdot \frac{L}{2}\right)\mathrm{d}S \\
&\quad + \iint_{\text{sphere}} \left(\epsilon_0 uB\frac{(2R+L)^2}{2R(R+L)}\frac{L}{2} + 3\epsilon_0 uB\cos\theta \cdot R\cos\theta\right)\mathrm{d}S
\end{aligned}$$

$$
\begin{aligned}
&\quad - \iint_{\text{sphere}} \left(\epsilon_0 u B \frac{(2R+L)^2}{2R(R+L)} R\cos\theta + 3\epsilon_0 u B \cos\theta \cdot \frac{L}{2} \right) \mathrm{d}S \Bigg] (-B\hat{\mathbf{x}}) \\
&= \left[2 \iint_{\text{sphere}} \left(\epsilon_0 u B \frac{(2R+L)^2}{2R(R+L)} \frac{L}{2} + 3\epsilon_0 u B \cos\theta \cdot R\cos\theta \right) \mathrm{d}S \right] (-B\hat{\mathbf{x}}) \\
&= -2\epsilon_0 u B^2 R \iint_{\text{sphere}} \left(\frac{L(2R+L)^2}{4R^2(R+L)} + 3\cos^2\theta \right) \mathrm{d}S \, \hat{\mathbf{x}} \\
&= -2\epsilon_0 u B^2 R \left(\frac{L(2R+L)^2}{4R^2(R+L)} \cdot 4\pi R^2 + 4\pi R^2 \right) \hat{\mathbf{x}} \\
&= -8\pi\epsilon_0 u B^2 R^3 \left(\frac{L(2R+L)^2}{4R^2(R+L)} + 1 \right) \hat{\mathbf{x}}.
\end{aligned}
$$

Therefore,

$$
\Delta \left(\sum_q q\mathbf{r} \times \mathbf{B} \right) = \left[-8\pi\epsilon_0 u B^2 R^3 \left(\frac{L(2R+L)^2}{4R^2(R+L)} + 1 \right) + 8\pi\epsilon_0 v B^2 R^3 \right] \hat{\mathbf{x}}.
$$

5 Solve for u using the conservation of $\sum_q q\mathbf{r} \times \mathbf{B} - \mathbf{p}$

We have

$$
\begin{aligned}
\Delta\mathbf{p} &= \Delta \left(\sum_q q\mathbf{r} \times \mathbf{B} \right) \\
m(u-v) &= -8\pi\epsilon_0 u B^2 R^3 \left(\frac{L(2R+L)^2}{4R^2(R+L)} + 1 \right) + 8\pi\epsilon_0 v B^2 R^3 \\
\left[m + 8\pi\epsilon_0 B^2 R^3 \left(\frac{L(2R+L)^2}{4R^2(R+L)} + 1 \right) \right] u &= \left(m + 8\pi\epsilon_0 B^2 R^3 \right) v \\
u &= v \frac{m + 8\pi\epsilon_0 B^2 R^3}{m + 8\pi\epsilon_0 B^2 R^3 \left(\frac{L(2R+L)^2}{4R^2(R+L)} + 1 \right)}.
\end{aligned}
$$

Substituting in the numerical values $v = 1\,\mathrm{m/s}, m = 0.15\,\mathrm{kg}, B = 2 \times 10^6\,\mathrm{T}, R = 0.1\,\mathrm{m}, L = 1\,\mathrm{m}$, and $\epsilon_0 = 8.854 \times 10^{-12}\,\mathrm{F/m}$, we get

$$
\boxed{u = 0.0345\,\mathrm{m/s}.}
$$

6 Appendix: Derivation of the charge distribution due to the image charge

As shown in Figure 7, a grounded spherical shell centered at O is in the vicinity of a charge $q_{1'} < 0$ at point B, where $OB = R/x$. The corresponding image charge is $q_2 = -xq_{1'} > 0$ at point A, where $OA = xR$. Consider the electric field $\mathbf{E}$ at point C on the shell, which will be parallel to OC. Let D on BC be such that $AD \parallel OC$.

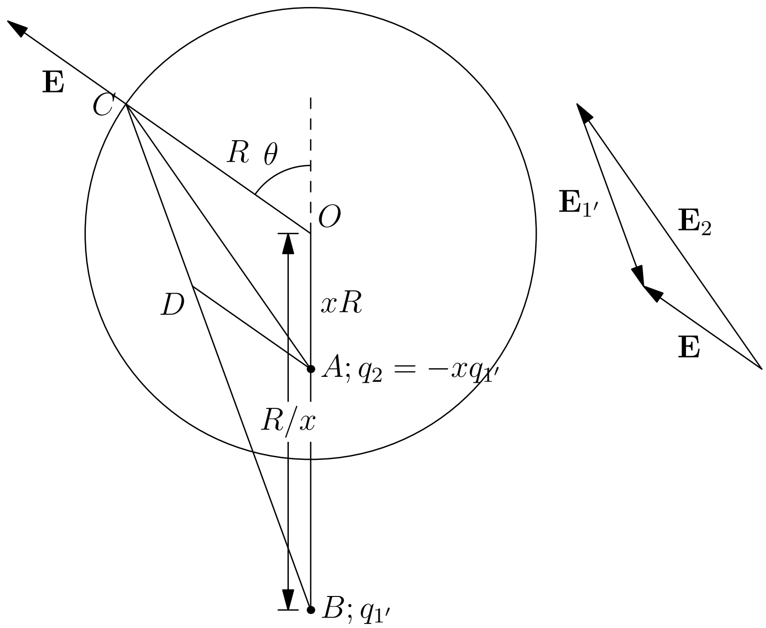

Figure 7: Induced charge distribution using image charges

Note that the electric field at C is due to $q_{1'}$ and q_2:

$$\mathbf{E} = \mathbf{E}_{1'} + \mathbf{E}_2,$$

where the vector triangle formed by $\mathbf{E}, \mathbf{E}_{1'}, \mathbf{E}_2$ is similar to $\triangle ACD$ (Figure 7). Therefore,

$$\begin{aligned}
E &= \frac{AD}{AC} E_2 \\
&= \frac{AD}{OC}\frac{OC}{AC}\frac{q_2}{4\pi\epsilon_0 AC^2} \\
&= \frac{AB}{OB}\frac{q_2 R}{4\pi\epsilon_0 AC^3} \\
&= \frac{\frac{1}{x} - x}{\frac{1}{x}}\frac{q_2}{4\pi\epsilon_0 R^2 \left(\frac{AC}{R}\right)^3} \\
&= \frac{q_2(1-x^2)}{4\pi\epsilon_0 R^2 \left(\frac{AC}{R}\right)^3} = \frac{-q_{1'}}{4\pi\epsilon R^2}\frac{x(1-x^2)}{(1+x^2+2x\cos\theta)^{\frac{3}{2}}} \\
\sigma_{1'\mathrm{T}} = \epsilon_0 E &= \frac{-q_{1'}}{4\pi R^2}\frac{x(1-x^2)}{(1+x^2+2x\cos\theta)^{\frac{3}{2}}}.
\end{aligned}$$