Physics Cup 2021, Problem 1

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Suppose at some instant the dumbbell deviates from its original orientation by angles α and β, as depicted in Fig. 1.Due to the Lorentz force acting on the electrons inside the dumbbell, the charge within it will be redistributed. Since $r \ll R$, we can neglect the charge distribution inside the rod and treat it like a wire connecting the two spheres. Moreover, since $L \gg R$, we can assume that spheres are far away from each other, so their potentials may be calculated as potentials of isolated spheres. The charges induced on the spheres will be $\pm q$, as shown in Fig. 1. The direction of Lorentz forces acting on the moving charged spheres will determine whether the system is stable with respect to either angular deviation. Let us first consider the angle β. When we look at the side-view of the system (Fig. 2), we see that the forces F (rather, their projections on the plane in question $F \sin\alpha$) will try to reduce the magnitude of the angle β, regardless of its sign. Therefore, the motion of the dumbbell is stable with respect to β, so we can put $\beta = 0$ for all subsequent calculations. As for the angle α, the forces will try to increase this angle (see the top view, Fig. 3), so the oscillations of the dumbbell will be in the horizontal plane.

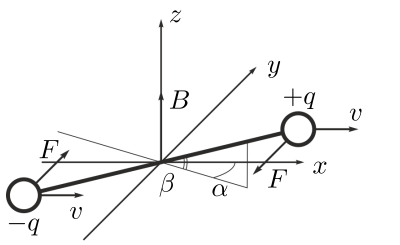

Fig. 1

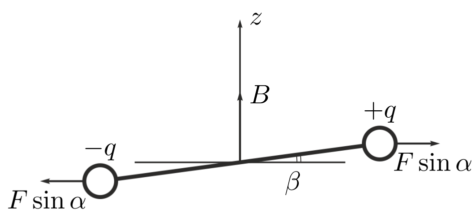

Fig. 2

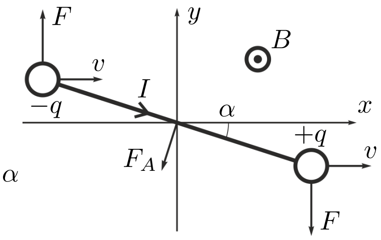

Fig. 3

Let us now derive the equations of the dumbbell's motion. First, we determine the charge q. To do that, we equate the work done by the Lorentz force along the rod with the potential difference of the spheres. Let v_{cx} and v_{cy} be the x- and y-components of the center-of-mass velocity. Rotation of the dumbbell does not contribute to the work, since these additional forces acting on the charges inside the rod have equal magnitudes but opposite directions at equal distances from the center and in the end their work will cancel out. Thus, the equation would be:

$$BL(v_{cy}\cos\alpha + v_{cx}\sin\alpha) = 2\,\frac{kq}{R},$$

where $k = 1/(4\pi\varepsilon_0)$. So the charge q is equal to

$$q = \frac{RBL}{2k}(v_{cy}\cos\alpha + v_{cx}\sin\alpha). \tag{1}$$

Let $\mathbf{r}$ be the position of the positive sphere with respect to the center, $\boldsymbol{\omega}$ the angular velocity of the rod. Using the formula

$$\mathbf{v} = \mathbf{v}_c + \boldsymbol{\omega}\times\mathbf{r}$$

we can write the equation for the center-of-mass motion:

$$m\dot{\mathbf{v}}_c = q(\mathbf{v}_c + \boldsymbol{\omega}\times\mathbf{r})\times\mathbf{B} - q(\mathbf{v}_c - \boldsymbol{\omega}\times\mathbf{r})\times\mathbf{B} + \mathbf{F}_A = 2q(\boldsymbol{\omega}\times\mathbf{r})\times\mathbf{B} + \mathbf{F}_A =$$

$$= 2q(\mathbf{r}(\mathbf{B}\cdot\boldsymbol{\omega}) - \boldsymbol{\omega}(\mathbf{B}\cdot\mathbf{r})) + \mathbf{F}_A = -2qB\dot{\alpha}\mathbf{r} + \mathbf{F}_A, \tag{2}$$

where $\mathbf{F}_A$ is the Ampere's force which is present due to the fact that the charge (1) changes over time, meaning that the current along the rod is

$$I = \frac{\mathrm{d}q}{\mathrm{d}t} = \frac{RBL}{2k}(\dot{v}_{cy}\cos\alpha - v_{cy}\sin\alpha\,\dot{\alpha} + \dot{v}_{cx}\sin\alpha + v_{cx}\cos\alpha\,\dot{\alpha}).$$

Hence the magnitude of the Ampere's force will be

$$F_A = IBL = \frac{RB^2L^2}{2k}(\dot{v}_{cy}\cos\alpha - v_{cy}\sin\alpha\,\dot{\alpha} + \dot{v}_{cx}\sin\alpha + v_{cx}\cos\alpha\,\dot{\alpha}). \tag{3}$$

Projecting (2) onto the x- and y-axes (see Fig. 3), using (3), we get for the x-component

$$m\dot{v}_{cx} = -\frac{RB^2L^2}{2k}(v_{cy}\cos\alpha + v_{cx}\sin\alpha)\cos\alpha\,\dot{\alpha} - \frac{RB^2L^2}{2k}(\dot{v}_{cy}\cos\alpha - v_{cy}\sin\alpha\,\dot{\alpha} + \dot{v}_{cx}\sin\alpha + v_{cx}\cos\alpha\,\dot{\alpha})\sin\alpha$$

and for the y-component

$$m\dot{v}_{cy} = \frac{RB^2L^2}{2k}(v_{cy}\cos\alpha + v_{cx}\sin\alpha)\sin\alpha\,\dot{\alpha} - \frac{RB^2L^2}{2k}(\dot{v}_{cy}\cos\alpha - v_{cy}\sin\alpha\,\dot{\alpha} + \dot{v}_{cx}\sin\alpha + v_{cx}\cos\alpha\,\dot{\alpha})\cos\alpha.$$

After simplifying these two expressions, we get the system of equations:

$$\dot{v}_{cx} = -A(\dot{v}_{cx}\sin^2\alpha + \dot{v}_{cy}\sin\alpha\cos\alpha + v_{cy}\cos 2\alpha\,\dot{\alpha} + v_{cx}\sin 2\alpha\,\dot{\alpha}); \tag{4}$$

$$\dot{v}_{cy} = -A(\dot{v}_{cx}\sin\alpha\cos\alpha + \dot{v}_{cy}\cos^2\alpha - v_{cy}\sin 2\alpha\,\dot{\alpha} + v_{cx}\cos 2\alpha\,\dot{\alpha}). \tag{5}$$

Here $A = RB^2L^2/(2mk)$. Note that the z-component of the velocity doesn't change, since there are no forces along this axis, therefore the whole motion of the dumbbell takes place inside the xy plane. To solve the system, we firstly multiply (4) by $\sin 2\alpha$, (5) by $\cos 2\alpha$ and add them together:

$$\dot{v}_{cx}\sin 2\alpha + \dot{v}_{cy}\cos 2\alpha = -A(\dot{v}_{cx}\sin\alpha\cos\alpha + \dot{v}_{cy}\cos^2\alpha + v_{cx}\dot{\alpha}). \tag{6}$$

Secondly, we multiply (4) by $\cos 2\alpha$, (5) by $\sin 2\alpha$ and subtract them from each other:

$$\dot{v}_{cx}\cos 2\alpha - \dot{v}_{cy}\sin 2\alpha = A(\dot{v}_{cx}\sin^2\alpha + \dot{v}_{cy}\sin\alpha\cos\alpha - v_{cy}\dot{\alpha}). \tag{7}$$

Rearranging the terms in (6) and (7), we arrive at a new system of equations:

$$\dot{v}_{cx}\sin 2\alpha(1 + A/2) + \dot{v}_{cy}\cos 2\alpha(1 + A/2) = -A/2\,\dot{v}_{cy} - Av_{cx}\dot{\alpha}; \tag{8}$$

$$\dot{v}_{cx}\cos 2\alpha(1 + A/2) - \dot{v}_{cy}\sin 2\alpha(1 + A/2) = A/2\,\dot{v}_{cx} - Av_{cy}\dot{\alpha}. \tag{9}$$

Now we do this procedure again: multiply (8) by $\sin 2\alpha$, (9) by $\cos 2\alpha$ and add them together, getting

$$\dot{v}_{cx}\left(1 + \frac{A}{2}\right) = \left(-\frac{A}{2}\dot{v}_{cy}\sin 2\alpha - Av_{cy}\cos 2\alpha\,\dot{\alpha}\right) + \left(-Av_{cx}\sin 2\alpha\,\dot{\alpha} + \frac{A}{2}\dot{v}_{cx}\cos 2\alpha\right). \tag{10}$$

Then we multiply (8) by $\cos 2\alpha$, (9) by $\sin 2\alpha$ and subtract them from each other:

$$\dot{v}_{cy}\left(1 + \frac{A}{2}\right) = \left(-\frac{A}{2}\dot{v}_{cy}\cos 2\alpha + Av_{cy}\sin 2\alpha\,\dot{\alpha}\right) + \left(-Av_{cx}\cos 2\alpha\,\dot{\alpha} - \frac{A}{2}\dot{v}_{cx}\sin 2\alpha\right). \tag{11}$$

Notice that now the equations (10)-(11) can be written in the form

$$\dot{v}_{cx}\left(1 + \frac{A}{2}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{A}{2}v_{cy}\sin 2\alpha\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{A}{2}v_{cx}\cos 2\alpha\right); \tag{12}$$

$$\dot{v}_{cy}\left(1 + \frac{A}{2}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{A}{2}v_{cy}\cos 2\alpha\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{A}{2}v_{cx}\sin 2\alpha\right). \tag{13}$$

Now we can integrate both (12) and (13) from $t = 0$ to the present moment, using the initial conditions

$$v_{cx}(0) = v, \quad v_{cy}(0) = 0, \quad \alpha(0) \approx 0$$

to get the integrals of motion:

$$(v_{cx} - v)\left(1 + \frac{A}{2}\right) = -\frac{A}{2}v_{cy}\sin 2\alpha + \frac{A}{2}(v_{cx}\cos 2\alpha - v);$$

$$v_{cy}\left(1 + \frac{A}{2}\right) = -\frac{A}{2}v_{cy}\cos 2\alpha - \frac{A}{2}v_{cx}\sin 2\alpha.$$

These equations can be rewritten into a system of *linear algebraic equations*:

$$v_{cx}(1 + A\sin^2\alpha) - v = -Av_{cy}\sin\alpha\cos\alpha;$$

$$v_{cy}(1 + A\cos^2\alpha) = -Av_{cx}\sin\alpha\cos\alpha;$$

the solution of which is

$$v_{cx} = v\,\frac{1 + A\cos^2\alpha}{1 + A}, \quad v_{cy} = -v\,\frac{A\sin\alpha\cos\alpha}{1 + A}. \tag{14}$$

Now we write the equation of rotational motion with respect to the center of the dumbbell. The Ampere's force exerts zero torque relative to the center, so the only two forces that change the angular momentum of the dumbbell are the Lorentz forces acting on the spheres. Hence the equation will be

$$\dot{\mathbf{L}} = q\mathbf{r} \times ((\mathbf{v}_c + \boldsymbol{\omega} \times \mathbf{r}) \times \mathbf{B}) - q(-\mathbf{r}) \times ((\mathbf{v}_c - \boldsymbol{\omega} \times \mathbf{r}) \times \mathbf{B}) = 2q\mathbf{r} \times (\mathbf{v}_c \times \mathbf{B}) =$$

$$= 2q(\mathbf{v}_c(\mathbf{r}\cdot\mathbf{B}) - \mathbf{B}(\mathbf{r}\cdot\mathbf{v}_c)) = -2q\mathbf{B}(\mathbf{r}\cdot\mathbf{v}_c) = -qL\mathbf{B}(v_{cx}\cos\alpha - v_{cy}\sin\alpha). \tag{15}$$

Let J be the moment of inertia of the dumbbell about the z-axis. Then, projecting (15) on the z-axis, we get

$$J\ddot{\alpha} = \frac{RB^2L^2}{2k}(v_{cy}\cos\alpha + v_{cx}\sin\alpha)(v_{cx}\cos\alpha - v_{cy}\sin\alpha).$$

Substituting $v_{cx},\ v_{cy}$ from (14) and simplifying, we get

$$J\ddot{\alpha} = \frac{RB^2L^2v^2}{4k(1+A)}\sin 2\alpha. \tag{16}$$

The stable equilibrium point for α is $\alpha = \pi/2$, since the right-hand side of (16) turns from positive to negative at that angle. Therefore, when the oscillations fully decay, the angle will be $\alpha = \pi/2$. Substituting this value into (14), we get for the terminal velocity

$$u_x = \frac{v}{1+A}, \quad u_y = 0,$$

giving the final answer

$$\boxed{u = \frac{v}{1+A} \approx \frac{mv}{2\pi\varepsilon_0 RB^2L^2} = 7\ \text{mm/s.}}$$