Let us work in the frame of reference of the centre of mass of the dumbbell.

Lemma 1

In the frame of reference of the centre of mass of the dumbbell, an induced electric field appears, and is given by

$$\mathbf{E_0} = \mathbf{v_0} \times \mathbf{B}, \tag{1}$$

where $\mathbf{v_0}$ is the velocity of the CM.

Proof. In the initial rest frame, the force on a particle of charge q and moving with velocity $\mathbf{v}$ is given by

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}. \tag{2}$$

In a frame moving with velocity $\mathbf{v}$, the magnetic field is unchanged, as long as the speeds are not relativistic; the force is, under the same constraint, also unchanged. However, an electric field *may* appear, so that the force may be written as

$$\mathbf{F} = q(\mathbf{E_0} + (\mathbf{v} - \mathbf{v_0}) \times \mathbf{B}). \tag{3}$$

Equality between the two forces implies that, for any $\mathbf{v}$,

$$\mathbf{E_0} + (\mathbf{v} - \mathbf{v_0}) \times \mathbf{B} = \mathbf{v} \times \mathbf{B}, \tag{4}$$

from which follows that

$$\mathbf{E_0} = \mathbf{v_0} \times \mathbf{B}, \tag{5}$$

as desired. □

In absolute value, since, in our situation, $\mathbf{v_0} \perp \mathbf{B}$ (i. e. the motion is in a plane perpendicular to $\mathbf{B}$, because the dumbbell initially has no velocity along the axis of $\mathbf{B}$ and the magnetic force does not act along that axis), we will have

$$E_0 = v_0 B. \tag{6}$$

In this CM frame, the induced electric field may cause a charging of the balls, in the following way: if the electric field has a component that acts along the line between the two balls (i. e. along the copper rod), there will be a net electromotive force acting in that way, that tends to create current flow from one of the balls to the other. Quickly enough, the current flow will stop due to the charging of the balls, that tends to counteract the electromotive force (i. e. as in an RC circuit). At all times, the charges of the balls - and hence their potentials - must be opposite, because the total charge of the system is zero. Let's see how quickly do the spheres reach electric equilibrium. We will first calculate the resistance of the wire (using the resistivity value for copper, which is $\rho \approx 0.0168\Omega\text{mm}^2\text{m}^{-1}$) and the capacitance of the system of the two spheres.

The resistance of the wire is

$$\mathcal{R} = \rho \frac{L}{\pi r^2} \approx 5.348\text{m}\Omega. \tag{7}$$

The capacitance of the system can be found as follows: the potential difference between an uncharged metal sphere and the charged second sphere is

$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R} - \frac{1}{R+L}\right), \tag{8}$$

where Q is the charge on the sphere. *Note:* This expression is approximate. However, it would be approximate anyway, because I have not taken into account the charge separation on the spheres themselves. Therefore, the denominator in the second term above (which may also be $1/L$, or $1/(2R+L)$, etc.) is approximately correct.

Since the potential difference between the spheres is $2V$ (as the spheres must have opposite charges, since the total charge of the system is 0), we see that

$$C = \frac{Q}{2V} = 2\pi\varepsilon_0 \frac{R(R+L)}{L} \approx 6.123 \cdot 10^{-12}\text{F}. \tag{9}$$

The time constant of this "RC circuit" is then

$$t = \sqrt{\mathcal{R}C} \approx 1.809 \cdot 10^{-7} s. \tag{10}$$

This is extremely small; therefore, we can safely assume that all charging processes happen essentially instantly, so that the charge on the spheres is always exactly enough to balance the induced electromotive force between them.

Let us further quantify this electromotive force. If we define $\mathbf{L}$ as a vector that points from one of the spheres to the other (let us denote one of them by sphere 1, and the other by sphere 2; this is purely arbitrary and of no consequence), then the electromotive force due to $\mathbf{E_0}$ is

$$\epsilon_{ind} = \int \mathbf{E_0} \cdot \mathbf{dl} = \mathbf{E_0} \cdot \mathbf{L} = E_0 L \cos\varphi, \tag{11}$$

where I have defined φ as the angle between $\mathbf{L}$ and $\mathbf{E_0}$. There may also be electromotive force due to the rotation of the dumbbell; it turns out that there isn't any net effect from it, but I won't bother to prove this as it won't affect us in the end.

We will have

$$Q = C\epsilon_{ind}, \tag{12}$$

as for a typical capacitor (this can also be seen from the relations above, with $\epsilon_{ind} = 2V$).

Now, what forces and what torques act on the system? Continuing to work in a frame of reference that moves along with the dumbbell, the following forces and torques arise:

- the *electric* force due to the field $\mathbf{E_0}$, which acts in opposite direction for the two balls, as their charges are opposite, and also has the same magnitude for both balls; this force usually provides torque (the two forces are opposite, but the torques due to them are *not* opposite, they are in the same direction), but no net force on the system. (*Note:* This force is magnetic, in the initial rest frame.)
- the magnetic forces due to the rotation of the two balls. We won't be bothered with them, in the end.
- the magnetic force on the rod, due to the current in it. This is the force that drives the translational motion of the dumbbell, but it provides no torque (it acts homogenously along the rod).

Let us study under what conditions may the rod be in equilibrium, as it is in the end. For it to be in equilibrium, there must be no torque and no net force on it. Furthermore, it may not rotate, because rotation would cause a change in the value of φ (see equation (11)), leading to net current through the rod, a net magnetic force, and hence loss of equilibrium.

If we consider the two balls stationary with respect to the CM, then the magnetic force on the rod - which is the only one that provides net force - is 0, for the following reason: if the balls do not rotate, then $\mathbf{L}$ is constant; the only way in which ϵ_{ind} could vary is through the variation of either $\mathbf{E_0}$ or φ (which can itself only vary if $\mathbf{E_0}$ varies, as long as $\mathbf{L}$ is constant); furthermore, looking at equation (1), $\mathbf{E_0}$ can only vary if $\mathbf{v_0}$ varies. In other words, the existence of a net force is dependent on the existence of acceleration. However, the acceleration is $\mathbf{0}$, as the rod is in equilibrium, so the net force is automatically $\mathbf{0}$.

We must only assure ourselves that the torque is balanced. Torque can only arise, in these conditions, from the *electric* forces on the two balls. For the torque to be zero, either the charge on the balls must be 0, so that the *electric* force is null, either the force arm of the *electric* forces is 0. ($\mathbf{E_0}$ cannot be null, because (1) implies that that would mean stopping of the dumbbell, which doesn't happen.) Let us study both situations separately.

- If $Q = 0$, that implies that (through (12)) $\epsilon_{ind} = 0$, or $\varphi = \pi/2$. Furthermore, because of (1), this implies that $\mathbf{v_0}$ is parallel to $\mathbf{L}$ (i. e. parallel to the rod), which is just the initial situation and corresponds to unstable equilibrium.
- If the *electric* force acts along the rod (so that its force arm is 0), that means that $\varphi = 0$. Again, from (1), this implies that the velocity $\mathbf{v_0}$ is perpendicular to the rod. This equilibrium must be the stable one.

We will therefore search for a situation in which the final speed is perpendicular to the final orientation of the rod. Furthermore, we see that, in this case,

$$\epsilon_{ind} = E_0 L. \tag{13}$$

Let us now move to the rest frame of the magnetic field and prove something else.

Lemma 2

While moving through an homogenous magnetic field, the generalized momentum $\mathbf{P}$ of a charged particle with constant charge q, given by

$$\mathbf{P} = m\mathbf{v} + q\mathbf{B} \times \mathbf{r}, \tag{14}$$

where m is its mass, $\mathbf{v}$ its velocity and $\mathbf{r}$ its momentary position vector, is conserved.

Proof. While moving through the magnetic field, using Newton's second law, we see that

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q\mathbf{v} \times \mathbf{B}. \tag{15}$$

Rearranging the terms and using the property of the cross product that $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$, we see that the previous equation implies

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + q\mathbf{B}\times\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = 0 \iff \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v} + q\mathbf{B} \times \mathbf{r}) = 0, \tag{16}$$

which is just what we wanted to prove. □

To use the conservation of electromagnetic momentum for the whole rod (the total electromagnetic momentum is just the sum of the momenta of the particles that make the rod up, which in our case can be considered just a juxtaposition of two charged particles, one with charge Q, the other with charge $-Q$), we need to prove just one more thing: that the charge transfer between the two "particles" does not modify $\mathbf{P}$. This will be proven later, after we will get an expression for this $\mathbf{P}$.

Let us define coordinate axes in the following way: at the initial time, the CM coincides with the centre of the Cartesian system, and the x-axis is the axis along which the dumbbell points and moves at this time. (The y-axis is also in the plane on which the magnetic field is perpendicular.) Since the entire dumbbell is uncharged at the initial time,

$$\mathbf{P} = m\mathbf{v} = mv\mathbf{e_x}, \tag{17}$$

where $\mathbf{e_x}$ is the unit vector in the x direction.

At the final time,

$$\mathbf{P} = m\mathbf{u} + Q\mathbf{B} \times \mathbf{r_2} + (-Q)\mathbf{B} \times \mathbf{r_1} \iff \mathbf{P} = m\mathbf{u} + Q\mathbf{B} \times (\mathbf{r_2} - \mathbf{r_1}). \tag{18}$$

But $\mathbf{r_2} - \mathbf{r_1} = \mathbf{L}$, by definition, so that

$$\mathbf{P} = m\mathbf{u} + Q\mathbf{B} \times \mathbf{L}. \tag{19}$$

We see here that, if Q changes by ΔQ, the change in the second term on the right is $\Delta Q\mathbf{B} \times \mathbf{L}$. However, the net impulse that acts on the rod is given by the magnetic force on the rod, which is

$$\Delta(m\mathbf{v}) = (I\mathbf{L} \times \mathbf{B})\Delta t = (I\Delta t)\mathbf{L} \times \mathbf{B} = \Delta Q\mathbf{L} \times \mathbf{B}, \tag{20}$$

which is just the opposite of the quantity above. Their sum is 0. Hence, my assertion has been proven: **P** indeed stays constant for the whole dumbbell along the duration of the motion.

Equating the two expressions of the generalized momentum, we see that

$$m\mathbf{v} = m\mathbf{u} + Q\mathbf{B} \times \mathbf{L}. \tag{21}$$

Let us now use what we said before. We require that, at the final time, the velocity of the CM of the rod, $\mathbf{u}$, be perpendicular to the rod itself. However, $\mathbf{B} \times \mathbf{L}$ is also perpendicular to the rod. Therefore, the two terms on the right of (21) are parallel between them. This means that each of them must also be parallel to $\mathbf{v}$; since $\mathbf{v}$ is along the x-axis, $\mathbf{u}$ *must also be along the x-axis.*

Using this information, (21) becomes a scalar equation, along the x-axis. The second term is just, in absolute value (since $\mathbf{B} \perp \mathbf{L}$), using equations (6), (12) and (13),

$$Q\mathbf{B} \times \mathbf{L} = (C\epsilon_{ind})BL = C(E_0 L)BL = C(uB)BL^2 = uC(BL)^2. \tag{22}$$

Therefore, (21) becomes

$$m(v-u) = uC(BL)^2 \Longrightarrow u = v\frac{m}{C(BL)^2 + m}. \tag{23}$$

Finally, substituting the expression for C,

$$\boxed{u = v\frac{mL}{2\pi\varepsilon_0 R(R+L)(BL)^2 + mL}}. \tag{24}$$

Numerically,

$$u \approx 6.09 \cdot 10^{-3}\text{ms}^{-1}. \tag{25}$$

Remark 1: In this solution, I have taken L to be the distance between the centres of the balls. If L is instead taken as the actual length of the rod, which only includes the distance between the two closest points on the surface of the spheres, then the distance between their centres becomes $L + 2R$; substituting this into the previous expression, we get

$$u = v\frac{mL}{2\pi\varepsilon_0 R(R+L)(B(L+2R))^2 + mL}, \tag{26}$$

which numerically evaluates to

$$u \approx 4.24 \cdot 10^{-3}\text{ms}^{-1}. \tag{27}$$

A difference would also appear in the calculation of the capacitance, but the difference is small (and I have actually used this second definition of L in its calculation already, that is why the mL in the expression above did not change when the definition of L was changed).

I have also neglected the polarization (i. e. the separation of charge due to the field $\mathbf{E_0}$) of the balls.

Remark 2: In the solution, I have not been careful to preserve the signs of the different cross products that finally make up the second term from the right of (21). However, I can validate the answer using the following argument. If we suppose we chose the parameters of the system carefully so that the two terms on the right of equation (21) are nearly perfectly equal, then, if the sign were wrong (i. e. would be $-$ instead of $+$), the result would be that the coefficient of u would become nearly zero, which would require an extremely large value of u - clearly violating energy conservation. Since there is only one physically possible sign for that term (i. e. the sign is well-defined) and we have seen that $-$ leads to physical issues, then the sign must be $+$.