Physics Cup 2021 Problem 1

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1 Solution

First, we will discuss the scenario qualitatively. The dumbbell is made of copper, which is a good conductor. At first, the dumbbell will be polarized due to the Hall effect. Any small perturbation from equilibrium will cause the dumbbell to have net charge on one sphere and the opposite charge on the other sphere, and so the magnetic field will apply a non-restoring torque on the dumbbell, implying that the equilibrium is unstable. The dumbbell will then start to oscillate in a complicated fashion. As copper is not a perfect conductor, the current induced in the dumbbell will dissipate energy, and eventually the dumbbell will settle in a stable equilibrium, which is when the dumbbell is moving in a direction perpendicular to its length. One can verify that this position is indeed stable, as the torque is restoring.

To solve the problem, we first prove the following fact:

Suppose there is a uniform magnetic field $\mathbf{B}$, and there are some charges moving in the field. Initially, they have charge distribution $\rho_1(\mathbf{r})$ and finally, they have charge distribution $\rho_2(\mathbf{r})$. Then the total impulse applied to the charges is given by

$$\left(\int \rho_2(\mathbf{r})\mathbf{r}d^3\mathbf{r} - \int \rho_1(\mathbf{r})\mathbf{r}d^3\mathbf{r}\right) \times \mathbf{B}$$

Note that this is just the difference in the dipole moments cross the magnetic field.

Proof: Suppose the charges are moving with current density $\mathbf{J}(\mathbf{r}, t)$ and charge distribution $\rho(\mathbf{r}, t)$. Then, the force applied to the charges is given by

$$\mathbf{F} = \int (\mathbf{J}(\mathbf{r}, t) \times \mathbf{B})d^3\mathbf{r} = \int \mathbf{J}(\mathbf{r}, t)d^3\mathbf{r} \times \mathbf{B}$$

$$= \left(\int -(\nabla \cdot \mathbf{J})\mathbf{r}d^3\mathbf{r} + \int (\mathbf{J} + (\nabla \cdot \mathbf{J})\mathbf{r})d^3\mathbf{r}\right) \times \mathbf{B}$$

$$= \left(\int \frac{\partial \rho}{\partial t}\mathbf{r}d^3\mathbf{r} + \int \begin{pmatrix} \nabla \cdot (x\mathbf{J}) \\ \nabla \cdot (y\mathbf{J}) \\ \nabla \cdot (z\mathbf{J}) \end{pmatrix} d^3\mathbf{r}\right) \times \mathbf{B}$$

$$= \left(\frac{d}{dt} \int \rho \mathbf{r} d^3\mathbf{r} + \begin{pmatrix} \oint (x\mathbf{J}) \cdot d\mathbf{S} \\ \oint (y\mathbf{J}) \cdot d\mathbf{S} \\ \oint (z\mathbf{J}) \cdot d\mathbf{S} \end{pmatrix} \right) \times \mathbf{B}$$

Note that the second term is zero because as long as the surface S is big enough, the current density $\mathbf{J}$ will vanish on that surface. Thus, we have that $\mathbf{F} = \frac{d}{dt}(\int \rho(\mathbf{r}, t)\mathbf{r} d^3\mathbf{r}) \times \mathbf{B}$. We integrate over time to get the desired result.

Now we apply this fact to the main problem. Let $\mathbf{v}$ be the initial velocity of the dumbbell. To find the dumbbell's initial dipole moment, we use a reference frame moving with the dumbbell. In this frame, the $\mathbf{B}$ field stays the same, and there is an additional electric field $\mathbf{E}_1 = \mathbf{v} \times \mathbf{B}$. Since the dumbbell is motionless in this frame, the dipole moment of the dumbbell will just be $\alpha_1 \mathbf{E}_1$, where α_1 is the polarizability of the dumbbell in the direction perpendicular to its length. Similarly, to find the dumbbell's final dipole moment, we use a reference frame moving with the dumbbell's final velocity, $\mathbf{u}$. We see that the dipole moment of the dumbbell is now $\alpha_2 \mathbf{E}_2$, where α_2 is the polarizability of the dumbbell in the direction parallel to its length, and $\mathbf{E}_2 = \mathbf{u} \times \mathbf{B}$.

Thus, since impulse is equal to the change in momentum of the dumbbell, we have $m(\mathbf{u} - \mathbf{v}) = ((\alpha_2 \mathbf{u} \times \mathbf{B} - \alpha_1 \mathbf{v} \times \mathbf{B}) \times \mathbf{B})$. Using the fact that $\mathbf{u}$ and $\mathbf{v}$ are both perpendicular to $\mathbf{B}$, we solve to get

$$\mathbf{u} = \frac{m + \alpha_1 B^2}{m + \alpha_2 B^2} \mathbf{v}$$

.

Now, it remains to find α_1 and α_2. First, we consider α_1. Suppose we put the dumbbell in an electric field $\mathbf{E}$ perpendicular to the dumbbell's length. Then the hollow spheres on the ends of the dumbbell will each be polarized. It is well known that a conducting sphere has polarizability $4\pi\epsilon_0 R^3$. Thus, each sphere will have dipole moment $4\pi\epsilon_0 R^3 E$, meaning the total dipole moment of the dumbbell is $8\pi\epsilon_0 R^3 E$, since the polarization of the thin rod can be neglected (which is of order $(\frac{r}{L})^2$), as well as the dipole-dipole interaction (which is of order $(\frac{R}{L})^3$). Therefore, $\alpha_1 = 8\pi\epsilon_0 R^3$.

Finally, we consider α_2. Suppose we put the dumbbell in an electric field $\mathbf{E}$ parallel to the dumbbell's length. Then, there will be a net charge on one hollow sphere and the opposite charge on the other sphere (the charge on the thin rod can be neglected because it has negligible capacitance, and the whole dumbbell is a equipotential surface). Let the charges on the spheres be Q and $-Q$. Then the potential difference between the spheres due to their charges alone must balance the potential difference due to the external electric field:

$$\frac{2Q}{4\pi\epsilon_0 R} = EL$$

. Thus, we have $Q = 2\pi\epsilon_0 RLE$, so the dumbbell's dipole moment is $QL = 2\pi\epsilon_0 RL^2 E$. Therefore, $\alpha_2 = 2\pi\epsilon_0 RL^2$.

Plugging everything back into our expression for **u** gives:

$$u = \frac{m + 8\pi\epsilon_0 R^3 B^2}{m + 2\pi\epsilon_0 R L^2 B^2} v$$

The numerical value for u is approximately $0.046 \frac{m}{s}$.