Physics Cup 2021 Problem 2

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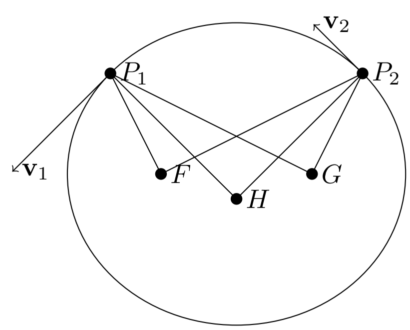
1 Solution

We assume the only force on the comet is the gravitational force from the Sun. It is known that the comet's trajectory must be a circle, ellipse, parabola, or hyperbola (in order of increasing eccentricity). The lowest eccentricity orbit, the circle, is clearly not possible because then the comet's speed would remain constant. Thus, we search for an elliptical orbit.

By Kepler's First Law, the Sun is at one of the foci of the comet's elliptical orbit. Let point F be the Sun, and G be the other focus point. Let a be the semi-major axis of the orbit and c be the distance between F and the center of the ellipse. Then the eccentricity e is $\frac{c}{a}$.



Note that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are tangential to the ellipse, and suppose P_1 is the location of the comet with velocity $\mathbf{v}_1$ and define P_2 similarly. Let the normal to the ellipse at P_1 intersect the normal to the ellipse at P_2 at point H. Let P_1 and P_2 vary along the ellipse such that the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$ is always 90 degrees (then the angle P_1HP_2 will be 90 degrees). Then the quantity $\frac{v_1}{v_2}$ attains its maximum value when $\frac{d(\frac{v_1}{v_2})}{d\theta} = 0$ or $\frac{dv_1}{v_1 d\theta} = \frac{dv_2}{v_2 d\theta}$, where $d\theta$ is a small change in angle of the velocity vectors (which are equal since the velocity vectors are always at the same angle to each other). Also, note that P_1 is closer to F than P_2 is. In general, if an object moves with velocity $\mathbf{v}$ and acceleration $\mathbf{a}$, it turns out that $\frac{dv}{v d\theta}$ equals the cotangent of the angle between $\mathbf{v}$ and $\mathbf{a}$. Since we know the acceleration of the comet is always directed towards F, we have that the angles FP_1H and FP_2H are equal, call this common angle α. Then by a property of ellipses, angles HP_1G and HP_2G are equal to α. In addition, since the angle P_1HP_2 is 90 degrees, we know that the angles P_1FP_2 and P_1GP_2 are both 90 degrees.

Note that by symmetry, we have that the line P_1P_2 is parallel to the line FG. Let r be the length FP_1. Then by a property of ellipses, $GP_1 = 2a - r$. By symmetry, $FP_2 = 2a - r$. Using angular momentum conservation, we get $v_1 r = v_2(2a - r)$, since the velocity vectors are at the same angle to the position vectors. According to the problem statement, we must have $\frac{v_1}{v_2} \geq 2$, but since we want to minimize e, we set $\frac{v_1}{v_2} = 2$. This means that $r = \frac{2}{3}a$. By the Pythagorean Theorem, we have $P_1P_2 = \frac{2\sqrt{5}}{3}a$. Note that FP_1P_2G forms an isosceles trapezoid. Also, by triangle similarity ($1 - 2 - \sqrt{5}$ right triangle) we can get $\frac{\sqrt{5}}{3}a - c = \frac{2}{3\sqrt{5}}a$, which gives $e = \frac{1}{\sqrt{5}}$ as our answer.