Physics Cup Problem 2

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The comet has an elliptical orbit. Denote the Sun - a focal point - by F_1, and the other focal point by F_2. The two points on orbit with $\vec{v}_1$ and $\vec{v}_2$ as velocities are denoted by M and N respectively. Draw the tangents to the ellipse at M and N. H and K are the projections of F_1 on the two tangents. (Figure 1).

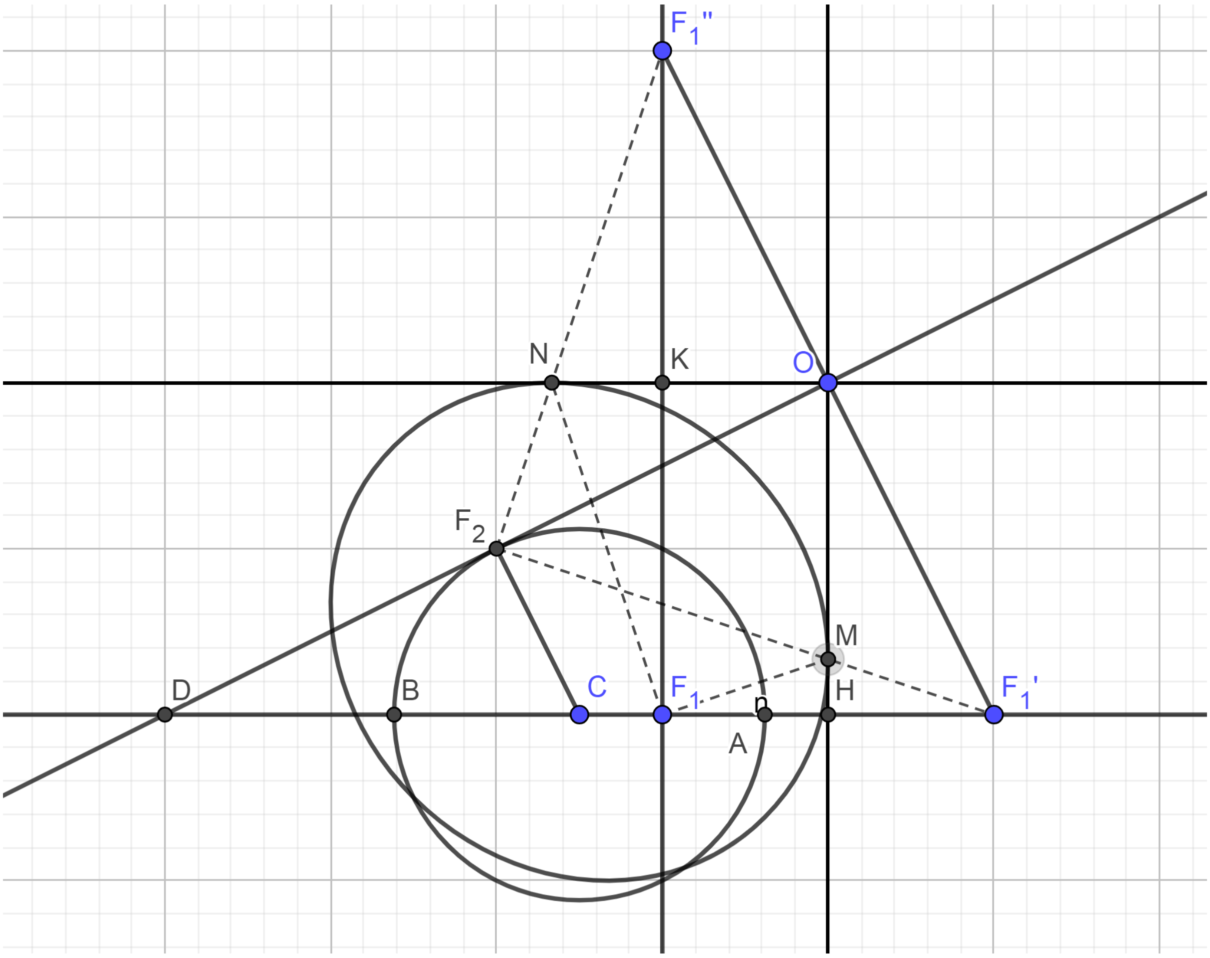

Figure 1: The completed construction of the problem

The conservation of angular momentum requires:

$$v_1 r_1 \sin(\vec{r}_1, \vec{v}_1) = v_2 r_2 \sin(\vec{r}_2, \vec{v}_2) \Longleftrightarrow v_1 \times F_1H = v_2 \times F_1K$$

Now we consider the two conditions of the problem:

$$\begin{cases} \vec{v}_1 \perp \vec{v}_2 & \iff OK \perp OH \\ \left|\vec{v}_1\right| = 2\left|\vec{v}_2\right| & \iff F_1H = \frac{1}{2}F_1K \end{cases}$$

Therefore, the problem is equivalent to a mathematical problem as follow (after setting $F_1H = 1$): Given the xOy plane and point $F_1(-1; -2)$, find the smallest possible eccentricity of an ellipse having F_1 as a focal point and touching the two axes.

To solve this problem, reflect F_1 with respect to the two axes, obtaining F_1' and F_1''. Because of the optical property of the ellipse (a light ray from F_1 reflecting on the ellipse would go through F_2, as per Fermat's principle), we can see that F_2, N, F_1'' are collinear, and so are F_2, M, F_1' (the dashed lines in Figure 1). It follows that $F_2F_1' = F_2F_1'' = 2a$, where a is the length of the semi-major axis. Therefore, F_2 lies on the perpendicular bisector (d) of $F_1'F_1''$. The position of F_2 on (d) will determine the eccentricity, as $e = \frac{c}{a} = \frac{F_2F_1}{F_2F_1'}$.

To find F_2, we consider the set of points with the same ratio of distances e to F_1 and F_1'. That set is an Apollonius circle with center C and radius R, and the circle must be on the same half-plane of edge OM with F_1 because F_2 must be so. According to the properties of the Apollonius circle, $R = \frac{2e}{1-e^2}$ and $CF_1' = \frac{2}{1-e^2}$, and circles with smaller e's are completely inside circles with larger e's (Figure 2).

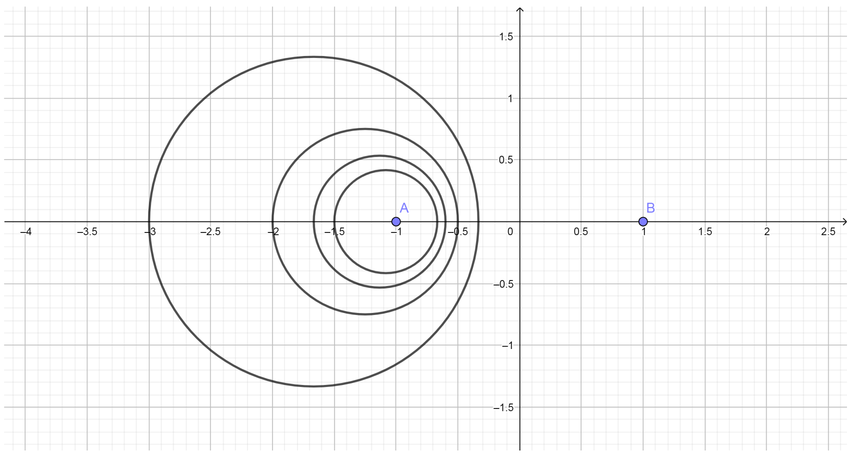

Figure 2: Apollonius circles of two points

Therefore, the circle with the smallest e that intersects (d) is the one that touches it. In other words, (d) is the tangent of that circle, and F_2 is the point of tangency.

Using Thales' theorem and similar triangles:

$$\begin{cases} \frac{CF_2}{CD} = \frac{OF_1'}{DF_1'} = \frac{1}{\sqrt{5}} \\ \frac{CF_2}{CD} = \frac{CF_2}{DF_1'-CF_1'} = \frac{R}{5-CF_1'} \end{cases}$$

$$\implies \sqrt{5}\frac{2e}{1-e^2} = 5 - \frac{2}{1-e^2}$$

$$\implies 5e^2 + 2\sqrt{5}e - 1 = 0$$

$$\implies e = \frac{1}{\sqrt{5}} \quad \text{where we pick the positive root}$$

Therefore, the minimum eccentricity is $e_{min} = \frac{1}{\sqrt{5}}$.