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Problem No 2

Let G denote the gravitational constant, M – mass of the sun (or another source of gravity), m – mass of the comet, $\vec{v}$ – its velocity, $\vec{L}$ –angular momentum of the system with respect to the sun, $\hat{r}$ – a unit vector directed from the sun to the comet. We'll restrict our consideration only to elliptical orbits. As we will see, it leads to a certain answer (minimal eccentricity, we are seeking) that is smaller than 1. It means that it is unnecessary to consider parabolic and hyperbolic orbits, as they have eccentricity greater than 1.

Lemma:

Laplace-Runge-Lenz vector defined as:

$$\vec{e} = \frac{1}{GMm}\left(\vec{v} \times \vec{L}\right) - \hat{r} \tag{1}$$

is constant in time and its magnitude is the eccentricity of the comet's orbit.

Proof:

Calculating time derivative of $\vec{e}$, we ascertain, that it is constant in time:

$$\frac{\mathrm{d}\vec{e}}{\mathrm{d}t} = \frac{1}{GMm}\left(\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \times \vec{L}\right) - \frac{\mathrm{d}\hat{r}}{\mathrm{d}t} \tag{2}$$

$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$ is equal to the acceleration of the comet. Using Newton's second law we can write $\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\frac{GM}{r^2}\hat{r}$. Let $\hat{\theta}$ denote a unit vector directed as $\vec{L} \times \hat{r}$. Let ω denote angular velocity of the comet's position relative to the sun. Then:

$$\frac{\mathrm{d}\vec{e}}{\mathrm{d}t} = \frac{1}{GMm}\frac{GML}{r^2}\hat{\theta} - \omega\hat{\theta} = 0 \tag{3}$$

Let $\hat{x}$ (fig. 1) denote a unit vector directed from the orbit's centre to the sun (or any direction if the orbit is circular). Let v_0 denote comet's velocity when it is closest to the sun (a distance d_0 from it). Vector $\vec{e}$ can be written as:

$$\vec{e} = \left(\frac{v_0 L}{GMm} - 1\right)\hat{x} = \left(\frac{d_0 v_0^2}{GM} - 1\right)\hat{x} \tag{4}$$

Let v_1 denote comet's velocity when it is furthest to the sun (a distance d_1 from it). Using the conservation of energy law and the conservation of angular momentum law, we can write:

$$\begin{cases} d_0 v_0 = d_1 v_1 \\ \frac{1}{2}mv_0^2 - \frac{GMm}{d_0} = \frac{1}{2}mv_1^2 - \frac{GMm}{d_1} \end{cases} \tag{5}$$

These two equations enable us to write v_1 and d_1 in terms of v_0 and d_0:

$$\begin{cases} v_1 = \left(\dfrac{2GM}{d_0 v_0^2} - 1\right) v_0 \\ d_1 = \dfrac{d_0}{\dfrac{2GM}{d_0 v_0^2} - 1} \end{cases} \tag{6}$$

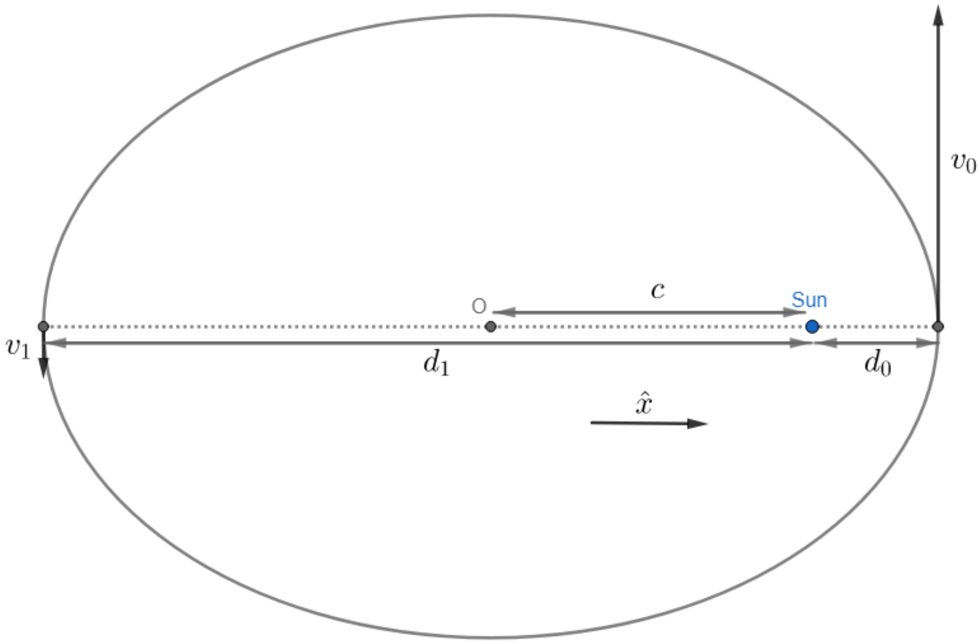

fig. 1

Major semi-axis a has length $\frac{d_0+d_1}{2}$. Thus $a = \frac{\frac{GM}{d_0 v_0^2}}{\frac{2GM}{d_0 v_0^2}-1} d_0$. Let c denote distance between the orbit's centre and the sun. $c = a - d_0$. Eccentricity of the orbit is defined as:

$$\frac{c}{a} = 1 - \frac{d_0}{a} = 1 - \frac{\dfrac{2GM}{d_0 v_0^2} - 1}{\dfrac{GM}{d_0 v_0^2}} = \frac{1 - \dfrac{GM}{d_0 v_0^2}}{\dfrac{GM}{d_0 v_0^2}} = \frac{d_0 v_0^2}{GM} - 1 \tag{7}$$

Comparing this result to (4), we see that $|\vec{e}| = \frac{c}{a}$. Therefore $\vec{e}$ is sometimes called the eccentricity vector. ■

Solution of the problem:

Let $\hat{r}_1$ denote a unit vector directed from the sun to the comet, when it has velocity $\vec{v}_1$. In analogous way we define $\hat{r}_2$. Condition (i) is equivalent to $(\vec{v}_1 \times \vec{L}) \cdot (\vec{v}_2 \times \vec{L}) = 0$. Condition (ii) is equivalent to $|\vec{v}_1 \times \vec{L}| = 2|\vec{v}_2 \times \vec{L}|$. Transforming (1) we get:

$$\vec{e} + \hat{r} = \frac{1}{GMm}\left(\vec{v} \times \vec{L}\right) \tag{8}$$

Using equation above we can express conditions (i) and (ii) as:

$$\begin{cases} (\vec{e} + \hat{r}_1) \cdot (\vec{e} + \hat{r}_2) = 0 \\ |\vec{e} + \hat{r}_1| = 2|\vec{e} + \hat{r}_2| \end{cases} \tag{9}$$

Let us analyse (9) geometrically.

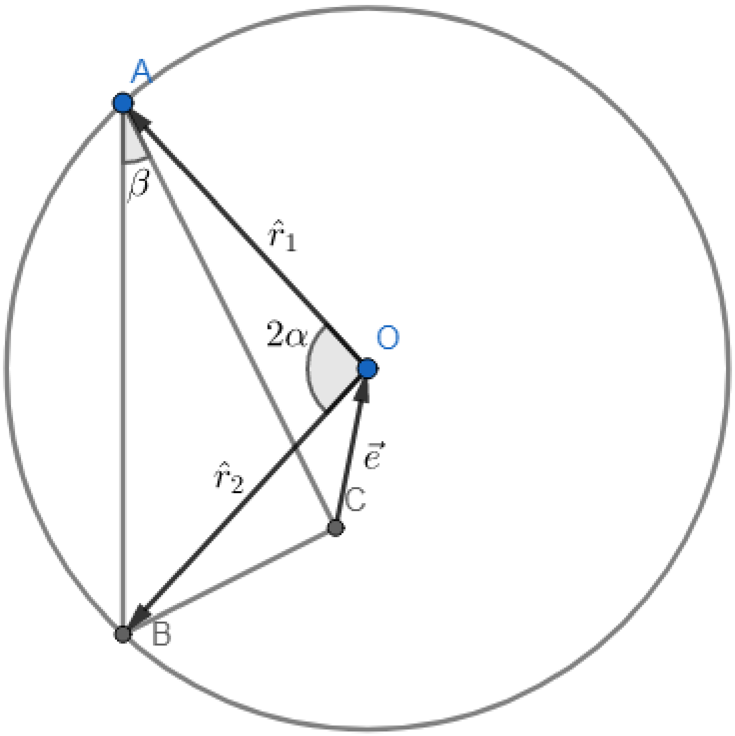

fig. 2

Let us consider a unit circle with centre O. Let A and B denote such points that $\overrightarrow{OA} = \hat{r}_1$ and $\overrightarrow{OB} = \hat{r}_2$. Let C denote such point that $\overrightarrow{CO} = \vec{e}$. Now simple vector addition says $\vec{e} + \hat{r}_1 = \overrightarrow{CO} + \overrightarrow{OA} = \overrightarrow{CA}$ and $\vec{e} + \hat{r}_2 = \overrightarrow{CO} + \overrightarrow{OB} = \overrightarrow{CB}$. Condition (i) says that ABC is a right-angled triangle. (ii) determines uniquely all its angles. $\beta = \sphericalangle BAC = \arctan\left(\frac{1}{2}\right)$. Let α denote a half of the angle BOA, so $\sphericalangle BOA = 2\alpha$. Thus $\sphericalangle BAO = \frac{\pi}{2} - \alpha$ and $\sphericalangle CAO = \left|\frac{\pi}{2} - \alpha \mp \beta\right|$. Sign $\mp$ should be taken negative, if C and O are on the same side of line AB. $\mp$ should be taken positive, if C and O are on the opposite sides of AB. Using cosine's theorem, we can write:

$$|\vec{e}|^2 = |AC|^2 + |AO|^2 - 2|AC| \cdot |AO| \cos(\sphericalangle CAO) \tag{10}$$

$|AC| = |AB| \cos\beta$ and $|AB| = 2|AO| \sin\alpha$. $|AO| = 1$, so:

$$|\vec{e}|^2 = |AB|^2 \cos^2\beta + 1 - 2|AB| \cos\beta \cos\left(\frac{\pi}{2} - \alpha \mp \beta\right) \tag{11}$$

$$|\vec{e}|^2 = 4\sin^2\alpha\cos^2\beta + 1 - 4\sin\alpha\cos\beta\sin(\alpha \pm \beta) \tag{12}$$

$$|\vec{e}|^2 = 4\sin^2\alpha\cos^2\beta + 1 - 4\sin\alpha\cos\beta\,(\sin\alpha\cos\beta \pm \sin\beta\cos\alpha) \tag{13}$$

$$|\vec{e}|^2 = 1 \mp \sin 2\alpha \sin 2\beta \tag{14}$$

Now we can see that the smallest possible value of $|\vec{e}|^2$ is at least $1 - \sin 2\beta$. In fact it is also the smallest possible value of $|\vec{e}|^2$. Once we found appropriate $\vec{e}$, $\hat{r}_1$ and $\hat{r}_2$, we can easily construct the orbit and two points on it, in which the comet has velocities $\vec{v}_1$ and $\vec{v}_2$ satisfying conditions (i) and (ii). Therefore:

$$e_{\min} = \sqrt{1 - \sin 2\beta} = \sqrt{1 - 2\tan\beta\cos^2\beta} = \sqrt{1 - \frac{2\tan\beta}{1 + \tan^2\beta}} \tag{15}$$

Applying $\beta = \arctan\left(\frac{1}{2}\right)$, we get finally:

$$e_{\min} = \frac{1}{\sqrt{5}} \tag{16}$$