Physics cup 2021, Problem 2

Fang Tzu, Hsu

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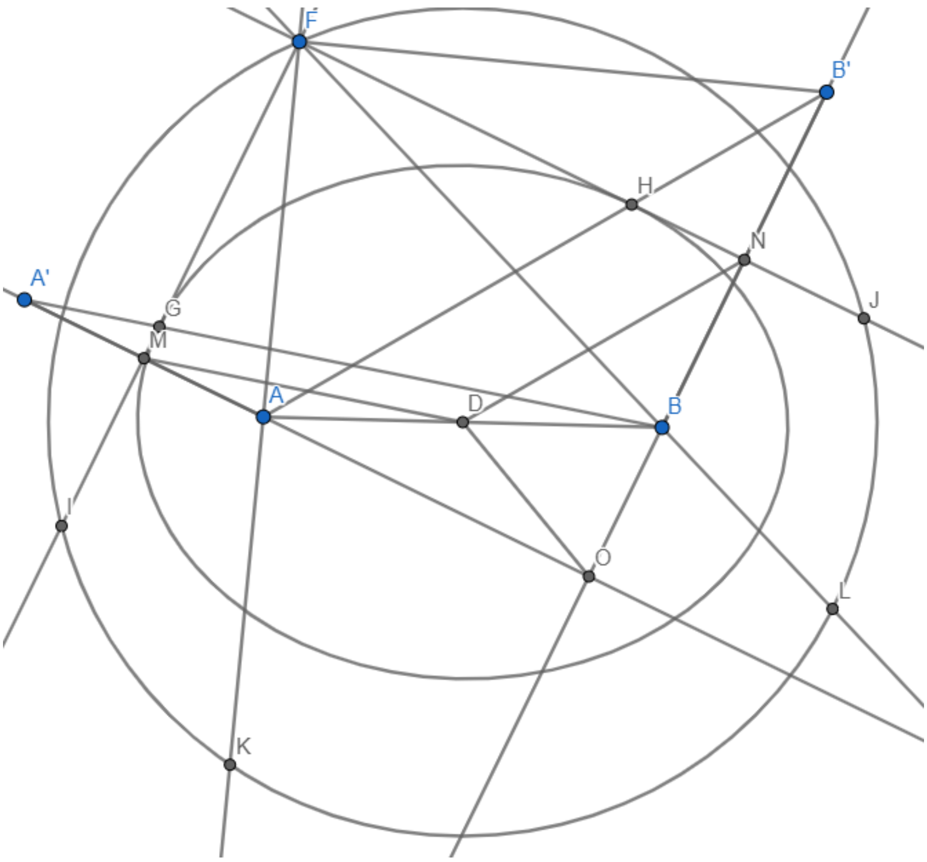
The shape of comet's orbit can be hyperbola, parabola, or ellipse. If we can find an ellipse orbit that matches the requirement, we don't have to consider the hyperbola/parabola orbit since the eccentricity of hyperbola and parabola is larger than that of ellipse.

Assume that the shape of the orbit is an ellipse. Consider the diagram below, A and B are two focal points of the ellipse, moreover A is the sun. The velocity of the comet at G and H are $\vec{v}_1$ and $\vec{v}_2$ respectively, and tangents of the ellipse at opint G and H meet at F. Since $\vec{v}_1 \perp \vec{v}_2$, $\angle$GFH = 90°. Let the width and height of the ellipse are $2a$ and $2b$ respectively. That is to say, AD = DB = $\sqrt{a^2 - b^2}$, where D is the midpoint of AB. For simplicity, let D is the origin of 2D xy coordinate, and the equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



Let's start from proving the following three useful lemmas that are related to the general property of ellipse.

Lemma 1:

No matter how we choose point G and point H (but remain ∠GFH = 90°), F lies on the circle $x^2 + y^2 = a^2 + b^2$. For simplicity, we call this circle ω in the following.

Proof:

Let A' and B' are the images of A and B reflect over the line FG and FH respectively. AA' intersects line FG at M, BB' intersects line FH at N. We know that ∠AGM = ∠BGF because of the optical property of the ellipse, and this implies that A',G,B are collinear. Similarly, B',H,A are collinear. MD = A'B/2 = (AG + GB) / 2 = a, similarly DN = a. Let AA' and BB' meet at O, since ∠AOB = 90°, DO = DB = $\sqrt{a^2 - b^2}$. Obviously $\mathrm{DF}^2 + \mathrm{DO}^2 = \mathrm{DM}^2 + \mathrm{DN}^2$, so $\mathrm{DF} = \sqrt{a^2 + b^2}$. This means that D lies on the circle of radius $\sqrt{a^2 + b^2}$ centered at D.

Lemma 2:

∠GFA = ∠BFN.

Proof:

$$\cos \angle \mathrm{FDA} = \frac{\mathrm{FD}^2 + \mathrm{DA}^2 - \mathrm{FA}^2}{2 \cdot \mathrm{FD} \cdot \mathrm{DA}} \tag{1}$$

$$\cos \angle \mathrm{FDB} = \frac{\mathrm{FD}^2 + \mathrm{DB}^2 - \mathrm{FB}^2}{2 \cdot \mathrm{FD} \cdot \mathrm{DB}} \tag{2}$$

Because $\angle \mathrm{FDA} + \angle \mathrm{FDB} = 180°$, $\cos \angle \mathrm{FDA} = -\cos \angle \mathrm{FDB}$, so we have

$$\mathrm{FA}^2 + \mathrm{FB}^2 = 2(\mathrm{FD}^2 + \mathrm{DA}^2) = 2[(a^2 + b^2) + (a^2 - b^2)] = (2a)^2 = \mathrm{AB'}^2 \tag{3}$$

But FB = FB', so $\mathrm{FA}^2 + \mathrm{FB'}^2 = \mathrm{AB'}^2$. This means that $\angle \mathrm{AFB'} = 90°$, so ∠GFA = ∠BFN.

Lemma 3:

∠GFA will get its smallest value when FD⊥AB.

Proof:

Pick one point F' on ω such that F'D⊥AB. Since the circumcircle of ΔF'AB lies inside circle ω, ∠AF'B≥ ∠AFB. From **lemma 2** we know that ∠GFA = $\frac{90° - \angle \mathrm{AFB}}{2}$, so ∠GFA will get its smallest value when $\angle \mathrm{AFB}$ is as large as possible, which occurs when FD⊥AB.

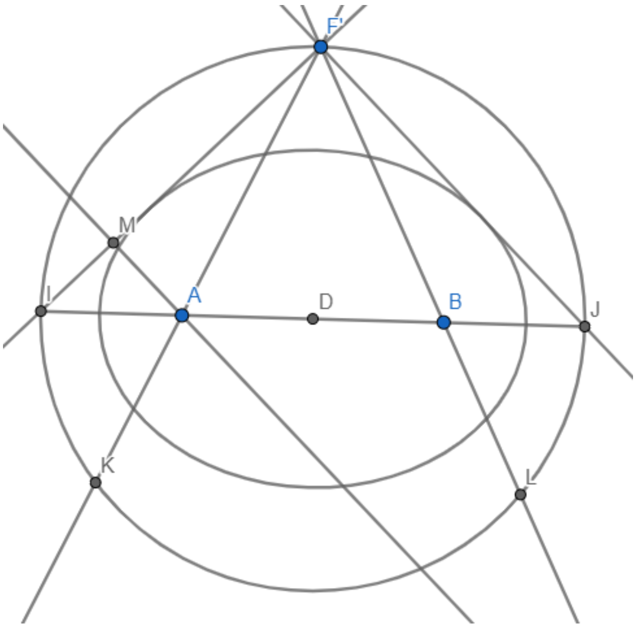
Back to the original question. Because the angular momentum is conserved while comet orbiting the sun, we have

$$\text{AM} \cdot |\vec{v}_1| = \text{MF} \cdot |\vec{v}_2| \tag{4}$$

Which is equivalent to

$$\tan \angle \text{GFA} = \frac{|\vec{v}_2|}{|\vec{v}_1|} = \frac{1}{2} \tag{5}$$

We want to construct an ellipse that makes us be able to find a point F on circle ω such that eqation (5) is satisfied. Without loss of generality, we only consider the situation which $\tan \angle \text{GFA} \leq 1$. We can deduce from simple observation that if the eccentricity of the ellipse is too small, the smallest possible value of $\tan \angle \text{GFA}$ is to large so that it cannot equal to 1/2. (The extreme case is that the orbit is a circle, than $\tan \angle \text{GFA}$ is always equal to 1.) So when the eccentricity of the ellipse is equal to its required smallest value, $\tan \angle \text{GF}'\text{A} = \frac{1}{2}$.



In this case IA/AJ = MA/MF' = 1/2, so we can let AD = DB = 1 = $\sqrt{a^2 - b^2}$ and DF' = 3 = $\sqrt{a^2 + b^2}$. Solving a and b we get $a = \sqrt{5}$ and $b = 2$. By definition, the eccentricity is $\frac{\sqrt{a^2-b^2}}{a} = \frac{1}{\sqrt{5}}$.