Physics Cup Problem 2

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February 2021

Because the question asks about the smallest value of eccentricity, it is legitimate to (only) consider elliptical orbit first. Figure 1 specifies the coordinate system and variable names used in this solution. Suppose the orbit of the comet takes the form of equation (1) in the polar coordinate system with one focus of the ellipse as its origin.

$$r = \frac{a(1-\epsilon^2)}{1-\epsilon\cos\theta} \tag{1}$$

where a is the length of semi-major axis and ϵ is eccentricity of the orbit.

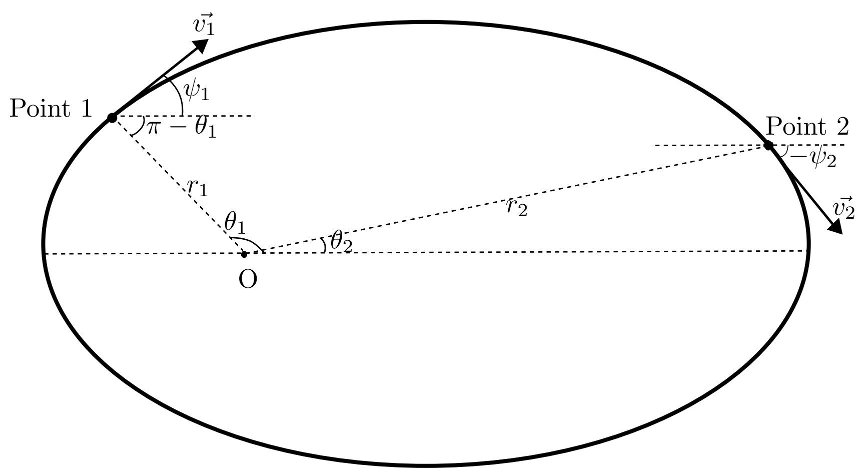

Figure 1: The Coordinate System and Variables

The velocity vector at every θ is parallel to the tangent line of the orbit at that point. The slope of it is calculated in

$$\tan\psi = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}(r\sin\theta)}{\mathrm{d}(r\cos\theta)} = \frac{\epsilon-\cos\theta}{\sin\theta} \tag{2}$$

Considering conservation of angular momentum at Point 1 and Point 2 gives

$$|\vec{v_1}||\vec{r_1}|\sin(\psi_1+\pi-\theta_1) = |\vec{v_2}||\vec{r_2}|\sin(\psi_2+\pi-\theta_2) \tag{3}$$

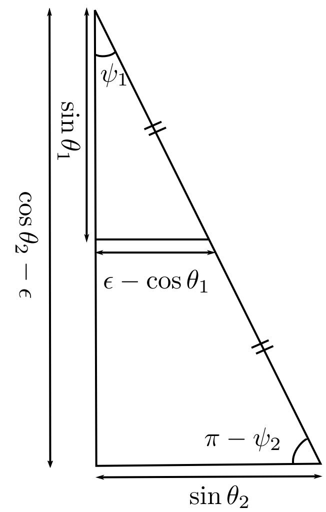

Figure 2: The Two Conditions displayed in diagram

Plugging Equation (1) and Equation (2) into Equation (3) gives

$$\frac{\sin\theta_2 \sec\psi_2}{\sin\theta_1 \sec\psi_1} = 2 \tag{4}$$

Because $\vec{v_1}$ and $\vec{v_2}$ are orthogonal, $\psi_2 - \psi_1 = \frac{\pi}{2}$. (It can be that $\psi_2 - \psi_1 = -\frac{\pi}{2}$, but due to symmetry, the two cases are equivalent) This condition and Equation (4) can be illustrated in Figure 2, which yields

$$\cos\theta_2 - \epsilon = 2\sin\theta_1 \tag{5}$$
$$2\epsilon - 2\cos\theta_1 = \sin\theta_2 \tag{6}$$

Using Lagrange multiplier to find the desired value of ϵ, we construct

$$F(\theta_1, \theta_2) = \cos\theta_2 - 2\sin\theta_1 + \lambda(2(\cos\theta_2 - 2\sin\theta_1) - 2\cos\theta_1 - \sin\theta_2) \tag{7}$$

and have

$$\frac{\partial F}{\partial \theta_1} = 0 \quad \frac{\partial F}{\partial \theta_2} = 0 \tag{8}$$
$$2(\cos\theta_2 - 2\sin\theta_1) - 2\cos\theta_1 - \sin\theta_2 = 0 \tag{9}$$

Under the constraint that $0 < \epsilon < 1$, we would have

$$\sin\theta_1 = \cos\theta_2 = -\frac{\sqrt{5}}{5} \tag{10}$$

and thus

$$\epsilon_{\min} = \frac{\sqrt{5}}{5} \tag{11}$$