Problem No 2

In a Centripetal Force Field $\overrightarrow{F} = -\frac{k}{r^2}\hat{r}$, there is a Runge-Lenz Vector $\overrightarrow{A} = \overrightarrow{p} \times \overrightarrow{L} - mk\hat{r}$ It is a Kepler laws.

$\because \overrightarrow{A} \cdot \overrightarrow{r} = (\overrightarrow{p} \times \overrightarrow{L}) \cdot \overrightarrow{r} - mk\overrightarrow{r} \cdot \hat{r}$

$\therefore Arcos\theta = \overrightarrow{L} \cdot (\overrightarrow{r} \times \overrightarrow{p}) - mk\overrightarrow{r}$

$r = \frac{L^2}{mk+Acos\theta}$

And the equation of the conic in polar coordinates $r = \frac{ep}{1+ecos\theta}$

So we get the eccentricity of the conic curve $e = \frac{A}{mk}$

We can make $\overrightarrow{e} = \frac{\overrightarrow{A}}{mk} = \frac{\overrightarrow{p} \times \overrightarrow{L}}{mk} - \hat{r}$

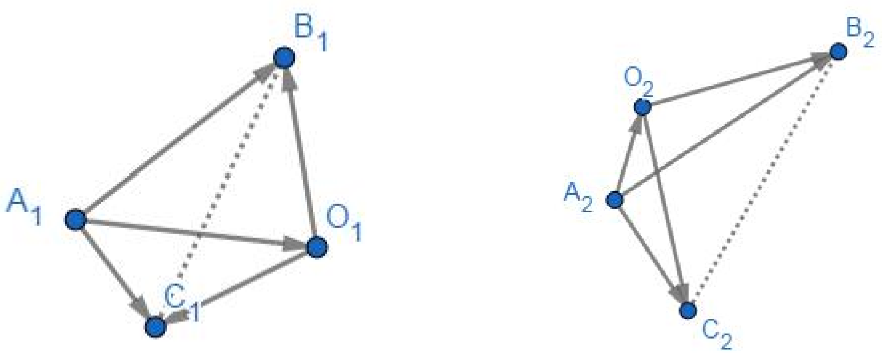
So we can write $\overrightarrow{e} = \frac{\overrightarrow{A}}{mk} = \frac{\overrightarrow{p_1} \times \overrightarrow{L}}{mk} - \hat{r_1}$ $\overrightarrow{e} = \frac{\overrightarrow{A}}{mk} = \frac{\overrightarrow{p_2} \times \overrightarrow{L}}{mk} - \hat{r_2}$

$\because \overrightarrow{v_1} \perp \overrightarrow{v_2}$, $|\overrightarrow{v_1}| = 2|\overrightarrow{v_2}|$ $\overrightarrow{L} \perp \overrightarrow{p}$

$\therefore (\overrightarrow{p_1} \times \overrightarrow{L}) \perp (\overrightarrow{p_2} \times \overrightarrow{L})$ $|\overrightarrow{p_1} \times \overrightarrow{L}| = 2|\overrightarrow{p_2} \times \overrightarrow{L}|$

For any a set of v_1 and v_2 that satisfy this condition , $\frac{\overrightarrow{p_1} \times \overrightarrow{L}}{mk}$, $\frac{\overrightarrow{p_2} \times \overrightarrow{L}}{mk}$, $\hat{r_1}$, $\hat{r_2}$, $\overrightarrow{e}$

These four vectors can form a quadrilateral.($\frac{\overrightarrow{p_1} \times \overrightarrow{L}}{mk} = \overrightarrow{AB}$, $\frac{\overrightarrow{p_2} \times \overrightarrow{L}}{mk} = \overrightarrow{AC}$, $\hat{r_1} = \overrightarrow{OB}$, $\hat{r_2} = \overrightarrow{OC}$, $\overrightarrow{e} = \overrightarrow{AO}$)



(Picture 1)

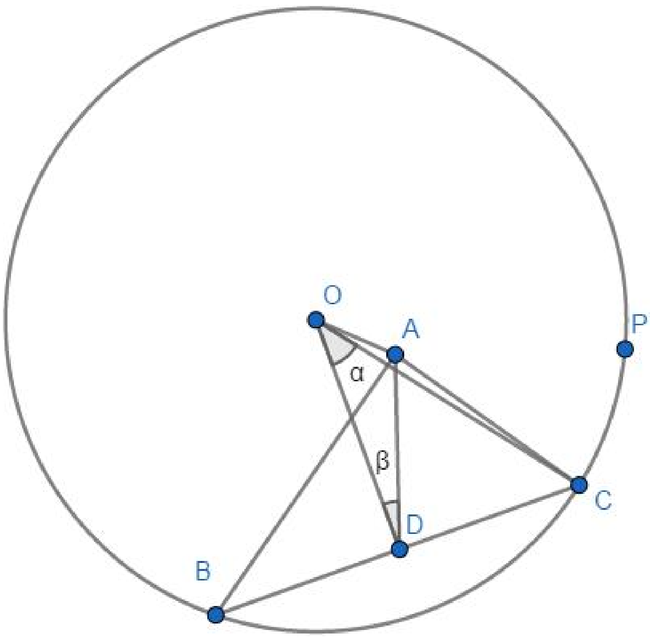
$|\hat{r1}| = |\hat{r2}| = 1$ $\overrightarrow{AB} \perp \overrightarrow{AC}$ $|\overrightarrow{AB}| = 2|\overrightarrow{AC}|$

First, draw a circle of radius 1 with center O . Second , let's take a chord and define it as BC.

Because we know the angle of triangle ABC, point A only has two points. (Picture 1)

$|\overrightarrow{AO}| = e$ Apparently , $|\overrightarrow{A_2O_2}| < |\overrightarrow{A_1O_1}|$ If I want to get the smallest eccentricity , I should take the point A_2

Third , let 's calculate $|\overrightarrow{AO}|$. Pass O point to make OD perpendicular to BC at point D. (Picture 2)



(Picture 2)

This is obtained by a simple geometric relationship. $|AD| = |CD| = sin\alpha$ $|OD| = cos\alpha$, $cos\beta = 4/5$

From the law of cosines,
$|AO| = \sqrt{|AD|^2 + |OD|^2 - 2|AD||OD|cos\beta} = \sqrt{1 - \frac{4}{5}sin2\alpha}$

when $2\alpha = \frac{\pi}{2}$ We can get the smallest eccentricity $e_{min} = \frac{\sqrt{5}}{5}$.