Physics Cup 2021 - Problem 2

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Without loss of generality, let our comet's orbit be the following ellipse, with central body located at $(e, 0)$. Note that e is the eccentricity.

$$x^2 + \frac{y^2}{1-e^2} = 1 \tag{1}$$

Denote the points of velocities $\overrightarrow{v_1}$, $\overrightarrow{v_2}$ by subscripts 2 and 1, respectively, reflecting that symmetry argument and the location of central body permits us to locate the points on the second and the first quadrant. Let r be the distance from the point on the ellipse to $(e, 0)$ and R be the distance to $(0,0)$.

Using the well-known vis-viva equation, $|\overrightarrow{v_1}| = 2|\overrightarrow{v_2}|$ yields

$$\frac{8}{r_2} - \frac{2}{r_1} = 3 \tag{2}$$

Apollonius's theorem applied to the triangle made of foci and comet point gives

$$R^2 = (r-1)^2 + (1-e^2) \tag{3}$$

We now exploit the orthogonality of $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$.

Proposition:

$$\frac{(r_1-1)}{\sqrt{e^2-(r_1-1)^2}} \frac{(r_2-1)}{\sqrt{e^2-(r_2-1)^2}} = -\frac{1}{1-e^2} \tag{4}$$

Proof:

First note that for our equation of ellipse, slope of the tangent line at (x_i, y_i) on the ellipse is $-(1-e^2)\frac{x_i}{y_i}$. Orthogonality of $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ gives

$$\left(-(1-e^2)\frac{x_1}{y_1}\right)\left(-(1-e^2)\frac{x_2}{y_2}\right) = -1 \tag{5}$$

Solve for the coordinates on the ellipse (1) satisfying $x^2 + y^2 = R^2$, from which we get

$$\left(\frac{x}{y}\right)^2 = \frac{(r-1)^2}{(1-e^2)(e^2-(r-1)^2)} \tag{6}$$

Substituting the expression (6) to (5) and noting $r_1 < 1 < r_2$, the proof is done.

□

For convenience, define a new variable: $l = r_1 - 1$, $l' = r_2 - 1$, which by (2) are related by

$$l' = \frac{5l + 3}{3l + 5} \tag{7}$$

Equation (4) becomes

$$\frac{l}{\sqrt{e^2 - l^2}} \frac{l'}{\sqrt{e^2 - l'^2}} = -\frac{1}{1 - e^2} \tag{8}$$

After some algebra, we can solve (8) for e^2.

$$e^2 = \frac{l^2 + l'^2 - 2l^2 l'^2}{1 - l^2 l'^2} \tag{9}$$

Finally, using (7) and (9), we get

$$e^2 = \frac{41l^4 + 30l^3 - 32l^2 - 30l - 9}{25l^4 + 30l^3 - 30l - 25} \tag{10}$$

This is minimized at $l = -\frac{1}{3}$, giving $\frac{1}{5}$, which is valid since $-1 < l < 0$.

Thus, we find the smallest possible eccentricity.

$$\boldsymbol{e = \frac{1}{\sqrt{5}}}$$

Further generalization:

We can solve the problem with the condition $|\overrightarrow{v_1}| = k|\overrightarrow{v_2}|$, where $k > 1$. $k < 1$ is also treated modulo relabeling. For $k = 1$, the answer is trivially $e = 0$ (circle).

Starting from modifying (2), we obtain the analog of (10).

$$e^2 = \frac{(k^4 + 6k^2 + 1)l^4 + 2(k^4 - 1)l^3 - 8k^2 l^2 - 2(k^4 - 1)l - (k^2 - 1)^2}{(k^2 + 1)^2 l^4 + 2(k^4 - 1)l^3 - 2(k^4 - 1)l - (k^2 + 1)^2} \tag{10 *}$$

This is minimized at $l = -\frac{k-1}{k+1}$, giving $\frac{(k-1)^2}{k^2+1}$.

Therefore, the smallest possible eccentricity is

$$\boldsymbol{e = \frac{k - 1}{\sqrt{k^2 + 1}}}$$