$$\vec{\varepsilon} = \hat{r} + \vec{L} \times \vec{v} = \text{const},$$

where $\vec{L}$ is a constant vector parallel to the angular momentum. Let us write this, using the requirements of the problem, in terms of complex numbers:

$$\varepsilon = \mathrm{e}^{\mathrm{i}\psi} + z = \mathrm{e}^{\mathrm{i}\phi} + 2z\mathrm{i}.$$

If we eliminate from here z, we obtain

$$\varepsilon - \mathrm{e}^{\mathrm{i}\psi} = \left(\varepsilon - \mathrm{e}^{\mathrm{i}\phi}\right)/2\mathrm{i}$$

so that

$$\varepsilon = \left(\mathrm{e}^{\mathrm{i}\psi} - \frac{\mathrm{e}^{\mathrm{i}\phi}}{2\mathrm{i}}\right)\left(1 - \frac{1}{2\mathrm{i}}\right)^{-1} = \mathrm{e}^{\mathrm{i}\psi}\left(1 - \frac{\mathrm{e}^{\mathrm{i}\varphi}}{2}\right)\left(1 - \frac{1}{2\mathrm{i}}\right)^{-1},$$

where $\varphi = \phi - \psi$. Here ψ can be always chosen so that RHS is real and the modulus of RHS is minimized with $\varphi = 0$. So minimal $\varepsilon = \frac{1}{2} \cdot \left(\frac{4}{5}\right)^{1/2} = 1/\sqrt{5}$.