Problem: At two different points in its orbit, a comet has velocities $\vec{v}_1$ and $\vec{v}_2$. If:

(i) $\vec{v}_1$ and $\vec{v}_2$ are orthogonal and

(ii) $|\vec{v}_1| = 2|\vec{v}_2|$,

what is the smallest possible eccentricity of the orbit?

Solution:

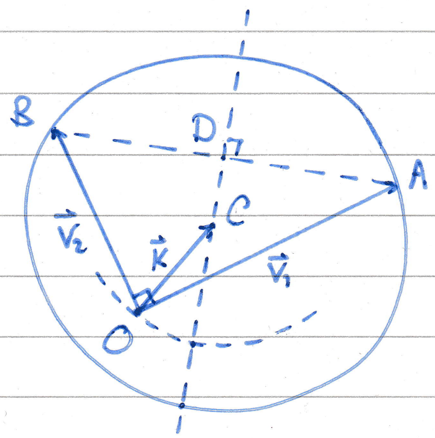
Claim 1: As the comet moves along its orbit, its velocity vector traces out a circle in velocity space.

Proof: It is well known that the Laplace-Runge-Lenz vector $\vec{A}$ is conserved, where:

$$\vec{A} = \vec{p} \times \vec{L} - GMm^2 \hat{r}. \qquad \text{(proof: differentiate this)}$$

It follows that $\vec{K} = \vec{v} + \dfrac{GMm(\hat{r} \times \vec{L})}{L^2}$ is conserved (since $\vec{A} = m\vec{K} \times \vec{L}$)

so $\vec{v}$ traces out a circle with centre $\vec{K}$ and radius $\dfrac{GMm}{L}$.



Given $\vec{v}_1$ and $\vec{v}_2$, the velocity circle is completely defined once $\vec{K}$ is given. Let the origin be O, the end of $\vec{v}_1$ be A, the end of $\vec{v}_2$ be B and the end of $\vec{K}$ (the centre of the velocity circle) be C (all vectors drawn from O).

C can be placed anywhere on the perpendicular bisector of AB.

Claim 2: $|\vec{K}| = \frac{GMme}{L}$, where e is the orbit's eccentricity.

Proof: $\vec{p} \times \vec{L} = \vec{A} + GMm^2 \hat{r}$

$$\therefore \underbrace{\vec{r} \cdot (\vec{p} \times \vec{L})}_{\substack{= \vec{L} \cdot (\vec{r} \times \vec{p}) \\ = L^2}} = \vec{r} \cdot \vec{A} + GMm^2 r = r\left(|\vec{A}|\cos(\theta) + GMm^2\right)$$

$$\therefore r = \frac{L^2}{GMm^2 + |\vec{A}|\cos(\theta)}$$

where θ is the angle between $\vec{r}$ and $\vec{A}$. This is the equation of a conic section with eccentricity

$$e = \frac{|\vec{A}|}{GMm^2} = \frac{L}{GMm}|\vec{K}| .$$

So, we want to minimise $e = \frac{|OC|}{|BC|}$.

Claim: The optimal position of C is on the circumcircle of OAB.

Proof: Using Ptolemy's inequality on the quadrilateral $OCAB$ gives:

$$|AB| \cdot |OC| + |BO| \cdot |AC| \geq |BC| \cdot |AO|$$

$$\begin{aligned} \therefore \frac{|OC|}{|BC|} = e &\geq \frac{|AO|}{|AB|} - \frac{|BO|}{|AB|} \cdot \frac{|AC|}{|BC|} \\ &= \frac{|AO| - |BO|}{|AB|} \\ &= \frac{V_1 - V_2}{\sqrt{V_1^2 + V_2^2}} \end{aligned}$$

We minimise e when we have equality, which occurs when OCAB is cyclic and the points are in order around the circle.

To get an elliptical orbit, we put C on the circumcircle of OAB at the point closer to O. Then

$$e = \frac{v_1 - v_2}{\sqrt{v_1^2 + v_2^2}} = \underline{\underline{\frac{1}{\sqrt{5}}}}.$$

Applying Ptolemy's inequality with the points in a different order gives an upper bound of $\frac{3}{\sqrt{5}}$ for e.

It is possible to use coordinate geometry instead of Ptolemy to minimise $\frac{|OC|}{|BC|} = e$.