Physics Cup 2021 Problem 2 Solution

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Before we begin, let us state the results that will be used in the solution, which are proven in the appendices:

I The velocities of the comet, when placed with their bases at the same point, form a circle (proof given in Appendix A)

II The ellipse formed by rotating the velocity vectors by 90°by their midpoints is similair to the ellipse of the trajectory (the precise way the ellipse is constructed as well as the proof is given in Appendix B)

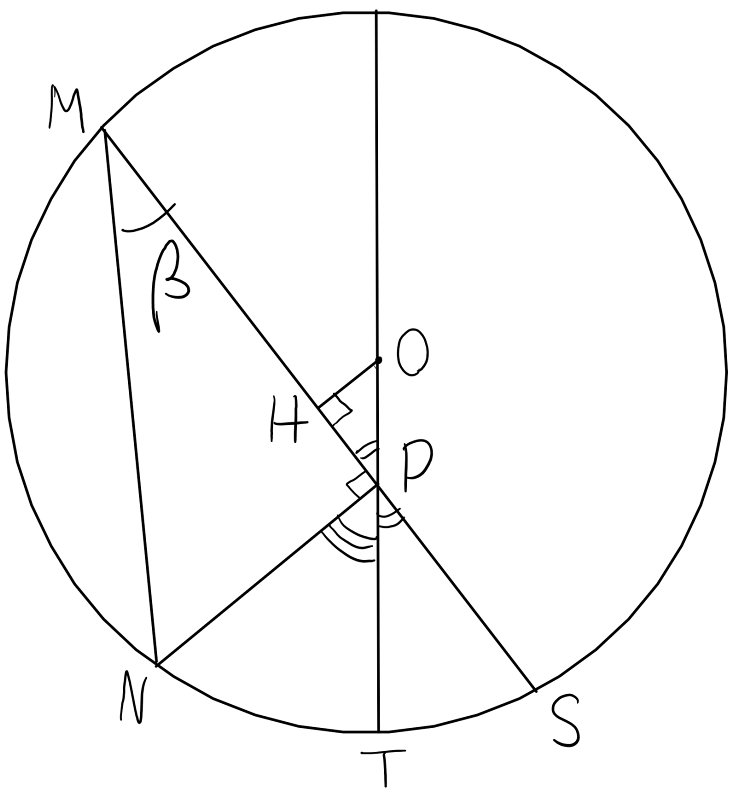

Figure 1: The construction of the maximum ratio between perpendicular velocity vectors (depicted as lines PN and PM). Point O is the center of the circle.

Let us consider circle formed by the velocity vectors (Figure 1), and draw perpendicular lines PM and PN from the base of velocity vectors P, which depict two perpendicular velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. We now seek to find the orientation of the lines so that their ratio PM/PN is maximum, or, equivalently, that PN/PM is minimum. We can do this by noting that the latter is just $\tan\beta$, so the minimum ratio of PN/PM is when the arc NTS is smallest. This corresponds to the symmetric case when the angles NPT=SPT=45°, since if we rotate the lines from this state (say, anti-clockwise), one arc (TS) will increase more than the other (TN) decreases, as the point on that side (S) is then further from P than the other point (N), and the same angle results in a larger arc length being covered when the distance is larger. Note that the symmetry also implies NP=PS in this case.

Now, let the distance OP decrease. Evidently, the arc NTS gets larger, so the ratio between the two velocites will decrease while staying the maximum possible of all pairs of perpendicular velocity vectors. When the eccentricity is smallest possible, the ratio must equal 2, or $\tan\beta = \frac{1}{2}$. Then, PM=2PN=2PS, and we can find the distance OP from

$$\frac{1}{2}\mathrm{MS} = \mathrm{MH} = \mathrm{HS} = \mathrm{PS} + \mathrm{OP}\cos(45^\circ) = \frac{1}{3}\mathrm{MS} + \frac{\sqrt{2}}{2}\mathrm{OP}$$

Hence, OP=$\frac{2}{6\sqrt{2}}\mathrm{MS} = \frac{1}{3\sqrt{2}}\mathrm{MS}$, or $\mathrm{MS} = 3\sqrt{2}\mathrm{OP}$. Note that MS is also related to OP by

$$\mathrm{MS}^2 = 2^2(\mathrm{OS}^2 - (\mathrm{OP}\sin 45)^2) = 4(R^2 - \frac{\mathrm{OP}^2}{2}) = 4R^2 - 2\mathrm{OP}^2$$

where R is the radius of the circle. Using $\mathrm{MS} = 3\sqrt{2}\mathrm{OP}$, we can express OP in terms of the radius R:

$$\mathrm{MS}^2 = 18\mathrm{OP}^2 = 4R^2 - 2\mathrm{OP}^2$$

$$20\mathrm{OP}^2 = 4R^2$$

$$\mathrm{OP} = \sqrt{\frac{1}{5}}R$$

Now, note that the semi-major axis of this ellipse is related to the radius of the circle as $2a = R$ (shown in Appendix B). Since the distance between the two foci O and P is just $d = 2ae$, the eccentricity is obtained as

$$e = \frac{d}{2a} = \frac{\mathrm{OP}}{R} = \sqrt{\frac{1}{5}}$$

As we have shown, this is the smallest eccentricity that will allow the required conditions to be satisfied.

Appendix A: Proving velocities of comet lie on a circle

We begin by analyzing two points on the comet's orbit, separated by a small angle $d\theta$ as viewed from the sun (Figure 2). The area the comet sweeps out is then $dA = \frac{1}{2}r^2 d\theta$. From Kepler's 2nd law, the time it takes for the comet to move between the two points is proportional to the area, or $dt \propto r^2 d\theta$. The velocity change in this time is given by

$$d\mathbf{v} = \frac{GMm}{r^2}\hat{\mathbf{r}}dt$$

or, $d\mathbf{v} \propto \frac{dt}{r^2}$, i.e. $d\mathbf{v} \propto \frac{r^2 d\theta}{r^2}$, that is, $d\mathbf{v}$ is directly proportional to the angle. If we split the whole trajectory into sectors of angle $d\theta$, the velocity change vector $d\mathbf{v}$ will have the same magnitude in each sector, but it will be rotated by an angle $d\theta$ with respect to its neighbors - this can be seen (Figure 2) by noticing that the force between neighboring points is rotated by $d\theta$, and, since it can be considered to be constant throughout each sector if $d\theta$ is small, the momentum change vector (and, hence, also the velocity change vector) is rotated by $d\theta$ with respect to momentum change vectors in neighboring sectors. Hence, if we were to place all velocity vectors with their bases at the same point, the velocity change vectors going from one vector to the neigboring one would form a regular polygon (see Figure 3). In the limit that $d\theta$ tends to 0, the polygon becomes a circle.

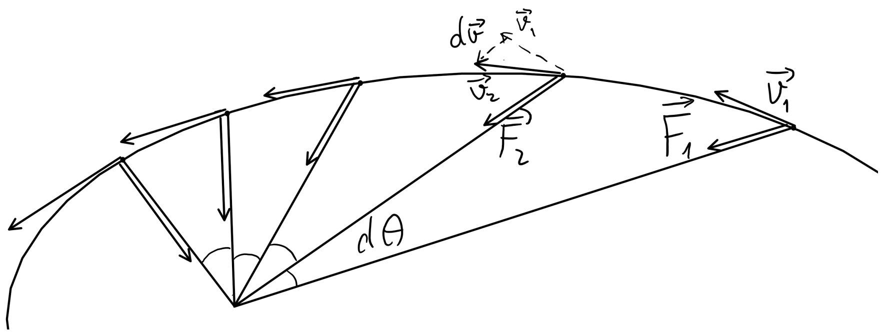

Figure 2: If the trajectory is split up into sections with angle $d\theta$ as viewed from the sun, the velocity change vector between neighboring points has the same magnitude, but is rotated by an angle $d\theta$ with respect to its neighboring velocity change vectors.

Appendix B: Proving similarity of ellipses

Let's start by rotating the vectors by 90 degrees anticlockwise, as well as rotating the whole circle 90 degrees anticlockwise. Now, if we draw the radius OA (see Figure 4) to the point where the non-rotated vector would have touched the circle and consider the intersect point B between the radius and the rotated vector, we can see that the triangles PCB and ACB are equal (CB is shared, the angles PCB and ACB are both straight and PC=AC by construction), so PB=AB. Notice that the sum OB+BP=OA is the radius of the circle, so it is the same for all intersects B constructed this way from the rotated vectors. Hence, all intersects B are part of an ellipse with O and P as its foci.

It is worth noting that the semi-major axis of the ellipse constructed this way has semi-major axis related to the radius of the velocity circle as $2a = R$, since, from Figure 4, we see that $2a = \mathrm{OB} + \mathrm{PB} = \mathrm{OB} + \mathrm{BA} = R$.

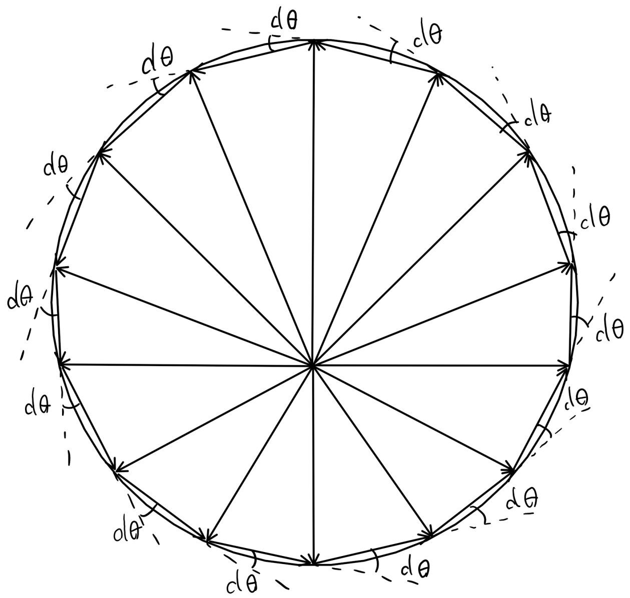

Figure 3: Placing the velocity vectors with their bases at the same point, the velocity change vectors going from one vector to its neighbor are of the same magnitude, but rotated by an angle $d\theta$ with respect to its neighbors. Hence, the velocity change vectors form a regular polygon, or, in the limit that $d\theta$ tends to 0, a circle.

To show that the ellipses are similar, let us start from the point where the shortest velocity vector touches the ellipse. If we also rotate the circle 90 degrees, it is clear the rotated velocity vector in the rotated circle is parallel to the velocity vector at that point in the original ellipse (Figure 5). Let us define this line in both ellipses as $\theta = 0$

If we now change θ by a small amount $d\theta$ (Figure 6), at the end of the change the velocities are again the same - indeed, as we already saw in Appendix A, a sector $d\theta$ in the ellipse of the trajectory corresponds to a velocity change that is described by an arc of angle $d\theta$ in the velocity circle, and, since the angle in the velocity ellipse is measured from the focus which is also the circle's center, the velocity changes by just this amount. We could then repeat the rotation to cover all points on both ellipses to conclude that the tangents of both ellipses coincide at all angles θ, so the two ellipses must be the same within a scale factor.

Finally, note that even with the scale factor difference between the ellipses, the eccentricity is the same: if we denote with a' the semi-major axis of the velocity ellipse, then $a = ca'$ is the semi-major axis of the trajectory ellipse, the distance between the two foci are $d' = 2a'e'$ and $d = 2ae$, respectively, where we for now distinguish between the two eccentricities. Ffrom the definition of the scale factor,

$$d = 2ae = cd' = 2ca'e' = 2ae'$$

Hence, e and e' must be the same.

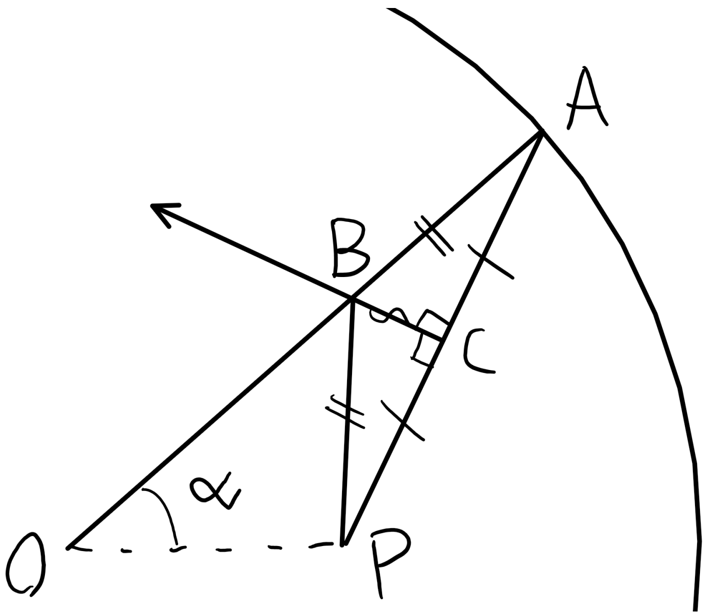

Figure 4: Each velocity vector on the circle defines a point B that belongs to an ellipse with foci O and P and semi-major axis given by $2a = R$.

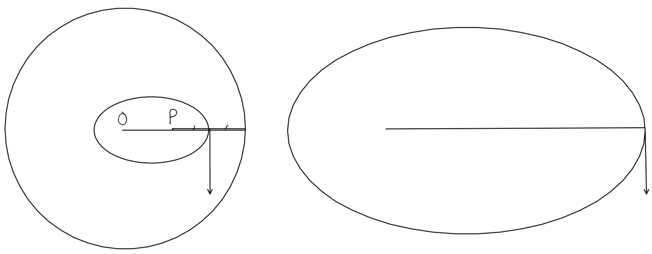

Figure 5: Velocity vectors in both ellipses are the same at the starting point (the velocity in the circle has been inverted to match the trajectory ellipse - this does not change the validity of the proof). Note that the starting circle was originally created by placing the velocity vectors as they would appear in the trajectory ellipse, i.e. they were placed at the same orientation.

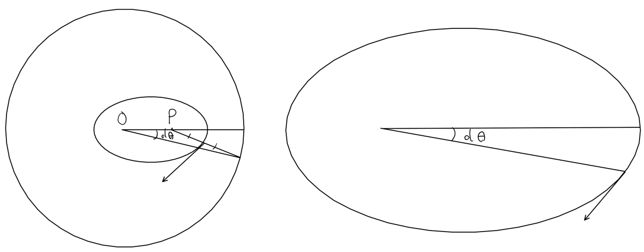

Figure 6: Rotating an angle $d\theta$ from the starting line, we see that the velocity at the corresponding points of both ellipses must be the same.