Problem 2, Physics Cup 2021

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1 Problem Statemet

At two different points in its orbit, a comet has velocities $\vec{v}_1$ and $\vec{v}_2$. If:
(i) $\vec{v}_1$ and $\vec{v}_2$ are orthogonal and
(ii) $|\vec{v}_1| = 2\,|\vec{v}_2|$, what is the smallest possible eccentricity of the orbit?

2 Solution

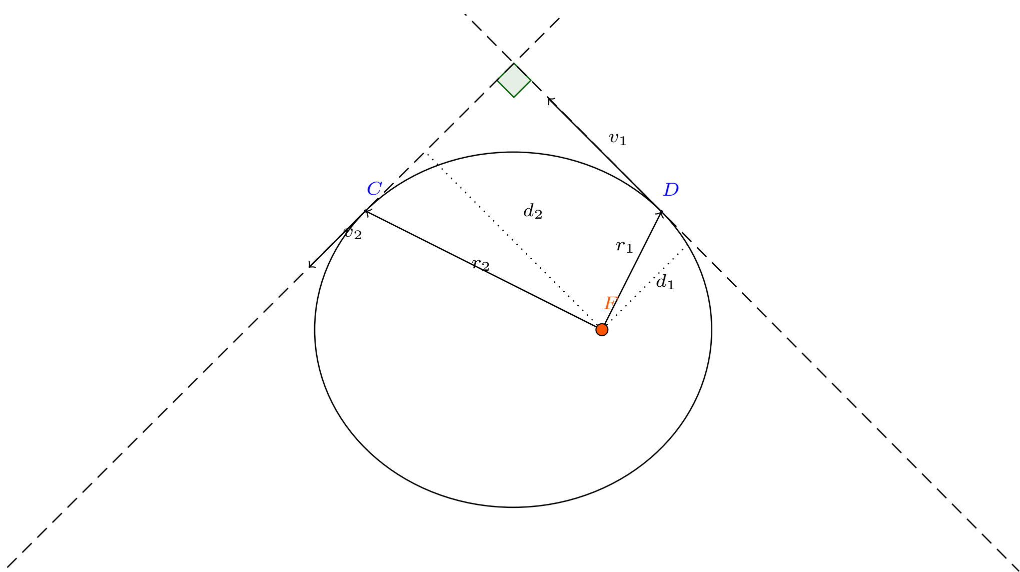

Figure 1: The Orbit

The orbit can not be a circle (eccentricity = 0) as the speed is constant in that case. The next candidate is an ellipse, which has an eccentricity $e \in (0, 1)$. The angular momentum remains constant for a central force. Hence

$$\vec{r}_1 \times \vec{v}_1 = \vec{r}_2 \times \vec{v}_2 \Rightarrow d_1 v_1 = d_2 v_2$$

where d_1 and d_2 are the respective perpendicular distances of the velocity vector lines from the focus (Fig. 1). Now, the condition $|\vec{v}_1| = 2\,|\vec{v}_2|$ yields us

$$d_2 = 2d_1$$

Therefore, the problem reduces to finding the smallest possible eccentricity of an ellipse with a pair of orthogonal tangents such that $d_2 = 2d_1$.

Lemma 2.1. *The locus of the intersection point of a pair of orthogonal tangents to an ellipse is a circle.*

Proof. The equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. We take a tangent to the ellipse from the point (h, k) with slope m. Then $(h^2 - a^2)m^2 - 2hkm + (k^2 - b^2) = 0$. This has two solutions m_1 and m_2 with $m_1 m_2 = \frac{k^2 - b^2}{h^2 - a^2}$, and the two tangents are perpendicular if $m_1 m_2 = -1$. Combining both conditions, we get $h^2 + k^2 = a^2 + b^2$, meaning that the locus is a circle of radius $\sqrt{a^2 + b^2}$.

$$x^2 + y^2 = a^2 + b^2$$

And its center coincides with the center of the ellipse $(0, 0)$. $\square$

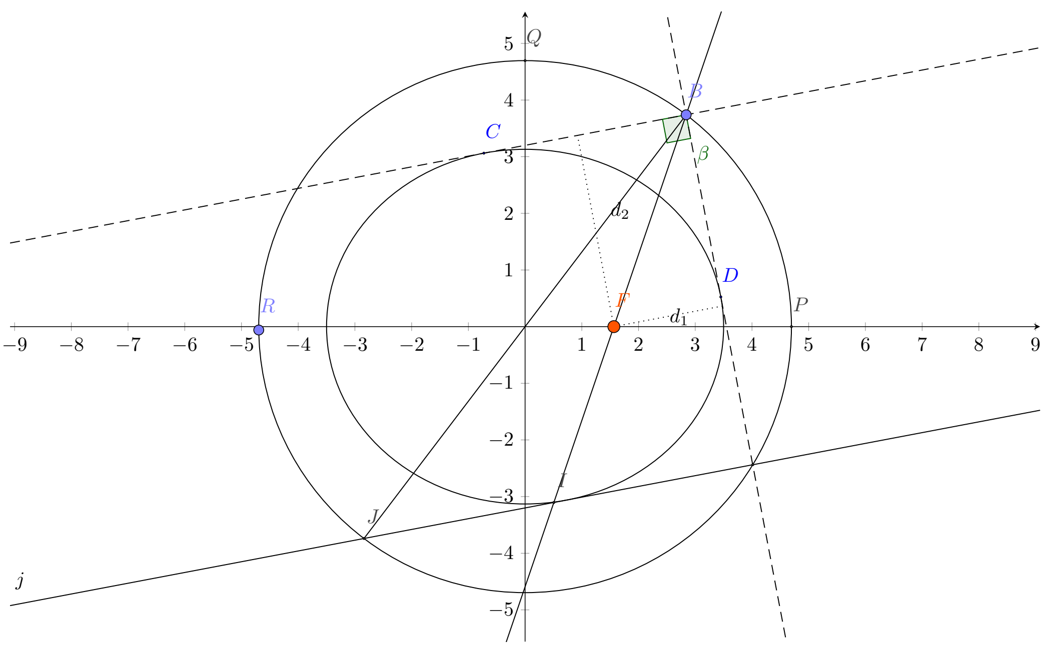

Figure 2: Ellipse with the locus circle

We now claim that the largest value of $\frac{d_2}{d_1}$ occurs when the intersection point of the pair of orthogonal tangents is at the co-vertex.
According to Fig. 2, $\tan\beta = \tan\angle FBD = \frac{d_1}{d_2}$. So, in order to maximize $\frac{d_2}{d_1}$, we

have to minimize $\tan\beta$ or β.
Due to symmetry, we can simply consider the case where B resides in the first quadrant. Also, $\beta = 45^\circ$ when $B = P$ and $B = R$. Therefore, following the Extreme Value Theorem and the symmetry of the system, we conclude that there must be an extremum of β at $B = Q$
If we follow the evolution of the segment IJ, we see that it increases (and β decreases) monotonously from $B = P$ to $B = Q$. Thus β can have an extremum only at $B = Q$, and it is a minimum.
WLOG, to make our calculations easy, we assume that $a = 1$. We know that $e^2 = 1 - \frac{b^2}{a^2}$, which implies $b^2 = 1 - e^2$. Now one tangent goes through $(0, \sqrt{2-e^2})$ and $(\sqrt{2-e^2}, 0)$; another one goes through $(0, \sqrt{2-e^2})$ and $(-\sqrt{2-e^2}, 0)$. Therefore, the equations of the tangents are

$$\begin{aligned} x + y - \sqrt{2-e^2} &= 0 \\ -x + y - \sqrt{2-e^2} &= 0 \end{aligned}$$

With the focus being $(e, 0)$, we calculate $\frac{d_2}{d_1}$

$$\left|\frac{-e-\sqrt{2-e^2}}{e-\sqrt{2-e^2}}\right| = 2 \Rightarrow \frac{-e-\sqrt{2-e^2}}{e-\sqrt{2-e^2}} = \pm 2$$

which gives one valid solution $\boxed{e = \frac{1}{\sqrt{5}}}$