Physics Cup Problem 3

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Note that a regular octagon looks like a very unresolved drawing of a circle. So let's see if we can get some inspiration from oscillation modes of a circular hoop. These modes are standing waves with wavelengths $\lambda_n = C/n$, where C is the circumference of the hoop. Oscillation modes of a regular octagon should be similar:

- $n = 1$ corresponds to translation (2 independent modes) and rotation (1 independent mode), which have a trivial frequency of 0.
- $n = 2$ corresponds to a roughly elliptical shape (2 independent modes) (Figure 1).
- $n = 3$ corresponds to deforming the octagon into a roughly triangular shape (2 independent modes) (Figure 2).
- $n = 4$ corresponds to deforming the octagon into a roughly square shape (1 independent mode) (Figure 3).
- $n \geq 5$ is impossible since the displacement of the corners of the octagon can only alternate between going out and going in at most 4 times.

The next three sections compute the oscillation frequencies corresponding to $n = 2, 3, 4$. In addition, we give the physical intuition behind the number of independent modes corresponding to each frequency. A more rigorous justification for these numbers is given in Section 4.

1 $n = 2$

All bars undergo translational motion: each blue bar is displaced by $x \ll l$ in a direction perpendicular to itself and each green bar is displaced by $x/\sqrt{2}$ in a direction parallel to itself. In addition, each green bar is rotated by an angle $\theta = \sqrt{2}x/l$ (Figure 1). Thus, the potential energy is

$$V = 8 \cdot \frac{1}{2}k\theta^2 = 4k\left(\frac{\sqrt{2}x}{l}\right)^2 = 8\frac{k}{l^2}x^2$$

and the kinetic energy is

$$T = 4 \cdot \frac{1}{2}m\dot{x}^2 + 4 \cdot \frac{1}{2}m\left(\frac{\dot{x}}{\sqrt{2}}\right)^2 + 4 \cdot \frac{1}{2} \cdot \frac{1}{12}ml^2\dot{\theta}^2 = 2m\dot{x}^2 + m\dot{x}^2 + \frac{1}{6}ml^2\frac{2\dot{x}^2}{l^2} = \frac{10}{3}m\dot{x}^2.$$

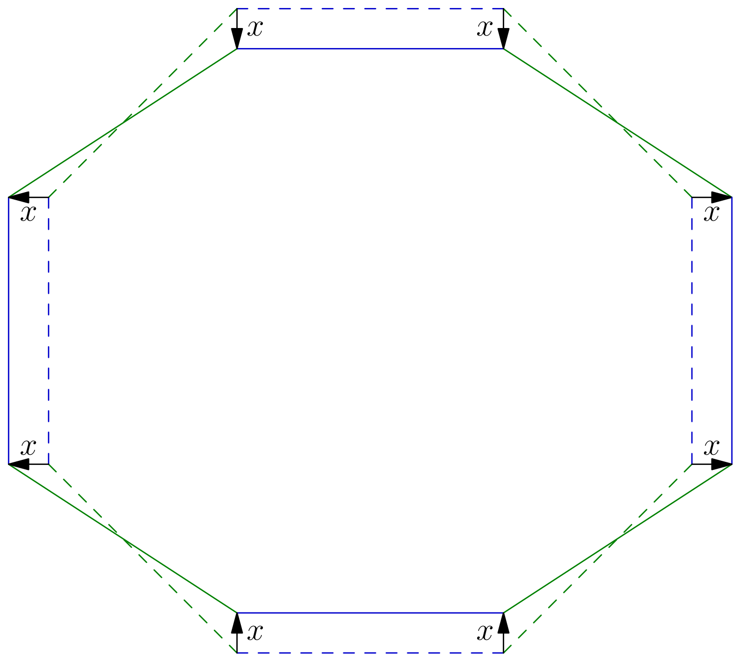

Figure 1: $n = 2$

The frequency of small oscillations is hence

$$f = \frac{1}{2\pi}\sqrt{\frac{8\frac{k}{l^2}}{\frac{10}{3}m}} = \frac{1}{2\pi}\sqrt{\frac{12}{5}\frac{k}{ml^2}}.$$

Since we may compress the octagon along any direction, there are two independent oscillation modes to allow for rotation. Two such modes can be, for example, the configuration in Figure 1 and the configuration rotated 45°. Then any other rotation angle rotation can be obtained from a linear combination of these two basis modes; for example, a rotation of 22.5° can be obtained by adding the two basis modes with equal amplitude.

2 $n = 3$

Here, the purple bars are translated downwards and the corners on the two sides swing upwards. Note that the shape of the octagon after half an oscillation period is obtained by simply reflecting across the horizontal. Therefore, by symmetry, the oscillation amplitudes of the two purple bars are equal. Assume that they have the same displacement of $x \ll l$ (Figure 2).

Since the center of mass of the octagon stays at rest, it should be displaced upwards by x in the reference frame of the purple bars (henceforth called the "purple frame"). It suffices to rotate the four green bars by an angle $\theta = 2\sqrt{2}x/l$ as shown in Figure 2. The two blue bars are then shifted upwards by $2x$ in the purple frame, and the COMs of the four green bars are displaced upwards by x; this causes the COM of the entire octagon to move upwards by $(2m \cdot 2x + 4m \cdot x)/(8m) = x$ in the purple frame.

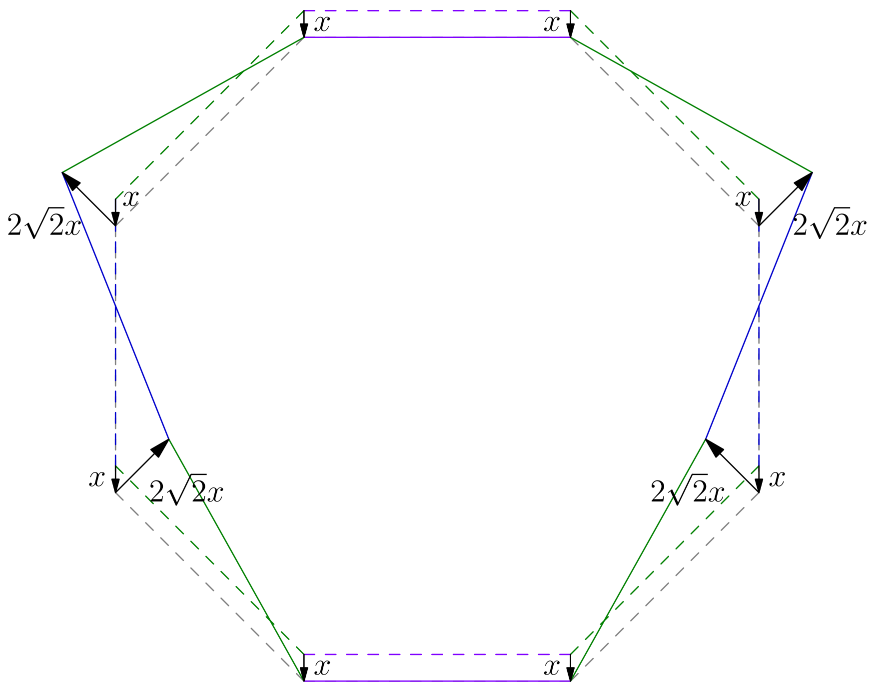

Figure 2: $n = 3$

The blue bars are rotated by $\varphi = 4x/l$, so the potential energy of the system is

$$V = 4 \cdot \frac{1}{2}k\theta^2 + 4 \cdot \frac{1}{2}k(\theta+\varphi)^2 = 2k\left(\frac{2\sqrt{2}x}{l}\right)^2 + 2k\left(\frac{\left(4+2\sqrt{2}\right)x}{l}\right)^2 = 32\left(2+\sqrt{2}\right)\frac{k}{l^2}x^2.$$

The kinetic energy as measured in the purple frame is

$$\begin{aligned} T' &= 4 \cdot \frac{1}{2} \cdot \frac{1}{3}ml^2\dot{\theta}^2 + 2 \cdot \frac{1}{2}m\left(2\dot{x}\right)^2 + 2 \cdot \frac{1}{2} \cdot \frac{1}{12}ml^2\dot{\varphi}^2 \\ &= \frac{2}{3}ml^2\frac{8\dot{x}^2}{l^2} + 4m\dot{x}^2 + \frac{1}{12}ml^2\frac{16\dot{x}^2}{l^2} \\ &= \frac{32}{3}m\dot{x}^2, \end{aligned}$$

so the kinetic energy in the rest frame (i.e., the COM frame) is

$$T = T' - \frac{1}{2}(8m)\dot{x}^2 = \frac{32}{3}m\dot{x}^2 - 4m\dot{x}^2 = \frac{20}{3}m\dot{x}^2.$$

So the frequency of oscillations is

$$f = \frac{1}{2\pi}\sqrt{\frac{32\left(2+\sqrt{2}\right)\frac{k}{l^2}}{\frac{20}{3}m}} = \sqrt{\frac{24}{5}\left(2+\sqrt{2}\right)\frac{k}{ml^2}}.$$

As with the $n = 2$ case, rotations are generated by two linearly independent oscillation modes with the frequency above. At first, it may seem reasonable that there are four independent modes corresponding to counter-clockwise rotation angles of $0°, 45°, 90°$ and $135°$. However, the $90°$ mode is actually a linear combination of the $0°$ and $45°$ modes. The displacements of the corners of the octagon for the $0°$ mode are, in counter-clockwise order starting from the left end of the top bar,

$$(0,-1), \quad (-2,1), \quad (2,1), \quad (0,-1), \quad (0,-1), \quad (-2,1), \quad (2,1), \quad (0,-1).^{1} \tag{1}$$

Similarly, the displacements for the $45°$ and $90°$ modes are

$$(1,-1), \quad (1,-1), \quad (-3,-1), \quad (1,3), \quad (1,-1), \quad (1,-1), \quad (-3,-1), \quad (1,3) \tag{2}$$

and

$$(-1,2), \quad (1,0), \quad (1,0), \quad (-1,-2), \quad (-1,2), \quad (1,0), \quad (1,0), \quad (-1,-2). \tag{3}$$

Since (1) + (2) + (3) $= \mathbf{0}$, the three modes are linearly dependent. Similarly, the $135°$ mode is a linear combination of the $45°$ and $90°$ modes and is therefore also expressible as a linear combination of the $0°$ and $45°$ modes. To conclude, there are, in fact, only two linearly independent oscillation modes for the $n = 3$ case.

3 $n = 4$

If every corner is displaced by $x \ll l$, then each bar is rotated by $\theta = 2x\cos\left(\frac{\pi}{8}\right)/l$ and translated by $x\sin\left(\frac{\pi}{8}\right)$ in a direction parallel to itself (Figure 3). Thus, the potential energy is

$$V = 8 \cdot \frac{1}{2}k(2\theta)^2 = 4k\left(\frac{4x\cos\left(\frac{\pi}{8}\right)}{l}\right)^2 = 64\cos^2\left(\frac{\pi}{8}\right)\frac{k}{l^2}x^2 = 16\left(2+\sqrt{2}\right)\frac{k}{l^2}x^2$$

and the kinetic energy is

$$\begin{aligned}
T &= 8\left[\frac{1}{2}m\left(\dot{x}\sin\left(\frac{\pi}{8}\right)\right)^2 + \frac{1}{2}\cdot\frac{1}{12}ml^2\dot{\theta}^2\right] \\
&= 4m\dot{x}^2\sin^2\left(\frac{\pi}{8}\right) + \frac{1}{3}ml^2\frac{4\dot{x}^2\cos^2\left(\frac{\pi}{8}\right)}{l^2} \\
&= \left[4\sin^2\left(\frac{\pi}{8}\right) + \frac{4}{3}\cos^2\left(\frac{\pi}{8}\right)\right]m\dot{x}^2 \\
&= \left(2-\sqrt{2}+\frac{2+\sqrt{2}}{3}\right)m\dot{x}^2 \\
&= \frac{8-2\sqrt{2}}{3}m\dot{x}^2.
\end{aligned}$$

[1] The factor of x is omitted as it is irrelevant.

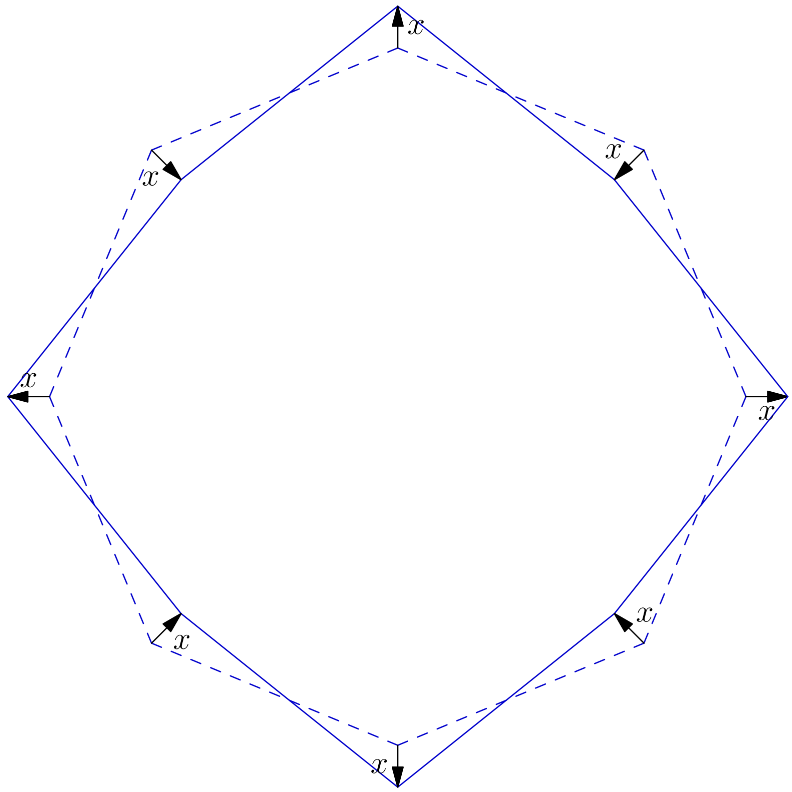

Figure 3: $n = 4$

The oscillation frequency is therefore

$$f = \frac{1}{2\pi}\sqrt{\frac{16\left(2+\sqrt{2}\right)\frac{k}{l^2}}{\frac{8-2\sqrt{2}}{3}m}} = \frac{1}{2\pi}\sqrt{24\frac{2+\sqrt{2}}{4-\sqrt{2}}\frac{k}{ml^2}} = \frac{1}{2\pi}\sqrt{\frac{24}{7}\left(5+3\sqrt{2}\right)\frac{k}{ml^2}}.$$

There can only be one oscillation mode with this frequency. For the perimeter of the octagon to alternate between going out and going in 4 times, the corners must translate the exact same way as shown in Figure 3. No rotation is possible. (One might think that rotation by 22.5° is possible, but this results in the exact same mode but with a sign flip in the displacements.)

4 A more rigorous justification for the number of independent modes corresponding to each frequency

The entire system has 8 degrees of freedom: there are 16 coordinates describing the positions of the corners, but there are 8 constraints as the distance between adjacent corners is fixed at l. So a vector $\mathbf{q}$ of 8 generalized coordinates completely describe the configuration of the system. If we define $\mathbf{q}$ such that the equilibrium point is at $\mathbf{q} = \mathbf{0}$, then the kinematics of the system near equilibrium is described by

$$\ddot{\mathbf{q}} = -A\mathbf{q},$$

where A is an 8×8 matrix. For an oscillation mode with angular frequency ω, $\ddot{\mathbf{q}} = -\omega^2 \mathbf{q}$, so

$$\omega^2 \mathbf{q} = A\mathbf{q}.$$

This an eigenvalue equation that describes all possible oscillation modes of the system: an eigenvector of A corresponds to the configuration of the oscillation mode, and the eigenvalue ω^2 is the square of the corresponding oscillation angular frequency. Since there are at most 8 linearly independent eigenvectors of A, there are at most 8 linearly independent oscillation modes.

Note that we have already found 8 linearly independent oscillation modes:

- 3 modes for $n = 1$ (2 translational modes and 1 rotational mode);
- 2 modes for $n = 2$ ($0°$ and $45°$ rotations of Figure 1);
- 2 modes for $n = 3$ ($0°$ and $45°$ rotations of Figure 2);
- 1 mode for $n = 4$ (Figure 3).

Thus, there can't be any oscillation modes that we've missed, and the oscillation modes described above constitute all possible oscillation modes of the octagon.