Physics Cup 2021 Problem 3

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1 Solution

Let the center of the octagon be O. Number the vertices of the octagon 1, 2, 3, 4, 5, 6, 7, 8 going clockwise. Number the bars 1, 2, 3, 4, 5, 6, 7, 8, where bar i is between vertices i and $i+1$. Define the z axis to be perpendicular to the plane of the octagon.

First, we find the number of degrees of freedom of the system for small oscillations. Naturally, we will constrain our system so that it has no net translational or rotational motion. Consider the vertices of the octagon. The state of the system is completely determined if we know the positions and velocities of the vertices. If the vertices are free to move, then they will have 16 degrees of freedom in total for their positions, and 16 for their velocities (2 for each vertex). However, we know the distance between consecutive vertices is constant since each bar's length doesn't change, so this applies 8 constraints to the positions and 8 to the velocities. Thus, we have 8 degrees of freedom left for the positions, and 8 for the velocities. Finally, the system shouldn't have net translational motion, which is 2 constraints for both position and velocity, and it shouldn't have net rotational motion, which is 1 constraint for both position and velocity. Thus, we are left with 5 degrees of freedom for position, and 5 for velocity. In total, the state of the system can be described by 10 (independent) parameters.

Next, we search for the normal modes of oscillation. Since the system is linear (this can be verified by simply writing down the force and torque equations for each bar), once we find the normal modes, we simply superpose them to get the general solution. The normal modes should be linearly independent, and when superposing them, each normal mode contributes 2 parameters (amplitude + phase), so it suffices to find 5 linearly independent normal modes. Once we do this, given any initial state of the system (which is 10 parameters), we can determine the amplitude and phase for each normal mode before superposing them to completely determine the evolution of the system.

The 1st normal mode, call it A, is where all the vertices oscillating radially, with the odd vertices oscillating at opposite the phase of the even vertices. To find the angular frequency of oscillation, we note that the state of the system can be described by the parameters θ and $\dot{\theta}$, where θ is the angular displacement of each bar (which are the same, by symmetry). The potential energy stored in each joint is $\frac{1}{2}k(\Delta\theta)^2$ where $\Delta\theta$ is the deviation of the joint angle from $\frac{3\pi}{4}$. Thus, since $\Delta\theta = 2\theta$, the total potential energy of the system is $8 * \frac{1}{2}k(2\theta)^2 = 16k\theta^2$. To find the kinetic energy of bar 1, for example, we note that velocity vectors of the ends of the bar are pointing along (or opposite of) the radial direction. Thus, the center of rotation P of bar 1 is the intersection of the line perpendicular to $O1$ at 1, and the line perpendicular to $O2$ at 2. The moment of inertia about P is $\frac{1}{12}ml^2 + m(\frac{l}{2}\tan\frac{\pi}{8})^2$ by the parallel axis theorem. In addition, the angular speed is $\dot{\theta}$. Thus, the kinetic energy of the bar is $\frac{1}{2}(\frac{1}{12}ml^2 + m(\frac{l}{2}\tan\frac{\pi}{8})^2)\dot{\theta}^2 = \frac{5-3\sqrt{2}}{12}ml^2\dot{\theta}^2$, so the total kinetic energy of the system is $\frac{10-6\sqrt{2}}{3}ml^2\dot{\theta}^2$. Finally, the angular frequency of the system is $\omega_A^2 = \frac{24}{5-3\sqrt{2}}\frac{k}{ml^2}$.

The 2nd normal mode, call it B, is where odd bars do not rotate, but oscillate radially. Note that bar 1 and bar 5 are exactly opposite phase with bar 3 and bar 7. In this case, we note that the state of the system can be described by the parameters x and $\dot{x}$, where x is the displacement of bar 1. Note that the angular displacement of bars 2, 4, 6, and 8 have magnitude $\frac{\sqrt{2}x}{l}$. Thus, the potential energy of the system is $8 * \frac{1}{2}k(\frac{\sqrt{2}x}{l})^2 = 8k\frac{x^2}{l^2}$. Note that bars 1, 3, 5, and 7 move at the same speed $\dot{x}$. Also, bars 2, 4, 6, 8 are rotating with angular speed $\frac{\sqrt{2}\dot{x}}{l}$ about a center of rotation $\frac{l}{2}$ away from the center of the bar. Thus, the total kinetic energy of the system is $4 * \frac{1}{2}m\dot{x}^2 + 4 * \frac{1}{2}(\frac{1}{12}ml^2 + m(\frac{l}{2})^2)(\frac{\sqrt{2}\dot{x}}{l})^2 = \frac{10}{3}m\dot{x}^2$. Finally, the angular frequency of the system is $\omega_B^2 = \frac{12}{5}\frac{k}{ml^2}$.

The 3rd normal mode, call it C, is where even bars do not rotate, but oscillate radially. Note that bar 2 and bar 6 are exactly opposite phase with bar 4 and bar 8. This case is equivalent to normal mode B, so the angular frequency is $\omega_C^2 = \frac{12}{5}\frac{k}{ml^2}$.

The 4th normal mode, call it D, is where bar 1 and bar 5 do not rotate, but oscillate while keeping the same displacement between each other. Bars 3 and 7 do not move radially, but rotate (in opposite directions) and move tangentially. In this case, the state of the system can be described by the parameters x and $\dot{x}$, where x is the displacement of bar 1. If bar 1 is displaced upwards by x, then so is bar 5, while bars 3 and 7 are displaced downwards by x, in order for the system's center of mass to remain at O. Note that bars 2, 4, 6, and 8 have no displacement in the vertical direction. The angular displacement of bars 2, 4, 6, and 8 are thus $\frac{2\sqrt{2}x}{l}$. The angular displacement of bars 3 and 7 are $\frac{4x}{l}$. Thus, the total potential energy of the system is $4 * \frac{1}{2}k(\frac{2\sqrt{2}x}{l})^2 + 4 * \frac{1}{2}k(\frac{2\sqrt{2}x}{l} + \frac{4x}{l})^2 = (64 + 32\sqrt{2})k\frac{x^2}{l^2}$. The kinetic energies of bars 1 and 5 are $\frac{1}{2}m\dot{x}^2$. The centers of bars 2, 4, 6, and 8 are moving at $\dot{x}$ and rotating with angular speed $\frac{2\sqrt{2}\dot{x}}{l}$. Thus, their kinetic energies are $\frac{1}{2}m\dot{x}^2 + \frac{1}{2} * \frac{1}{12}ml^2(\frac{2\sqrt{2}\dot{x}}{l})^2 = \frac{5}{6}m\dot{x}^2$. Finally, the kinetic energies of bars 3 and 7 are $\frac{1}{2}m\dot{x}^2 + \frac{1}{2} * \frac{1}{12}ml^2(\frac{4\dot{x}}{l})^2 = \frac{7}{6}m\dot{x}^2$. In total, the kinetic energy of the system is $2 * \frac{1}{2}m\dot{x}^2 + 4 * \frac{5}{6}m\dot{x}^2 + 2 * \frac{7}{6}m\dot{x}^2 = \frac{20}{3}m\dot{x}^2$. Therefore, the angular frequency of the system is $\omega_D^2 = \frac{48+24\sqrt{2}}{5}\frac{k}{ml^2}$.

The 5th normal mode, call it E, is where bar 3 and bar 7 do not rotate, but oscillate while keeping the same displacement between each other. Bars

1 and 5 do not move radially, but rotate (in opposite directions) and move tangentially. This case is equivalent to normal mode D, so the angular frequency is $\omega_E^2 = \frac{48+24\sqrt{2}}{5}\frac{k}{ml^2}$.

These normal modes can be shown to actually work by using symmetry or by finding explicitly the forces and torques between bars that produce the desired motion.

Now it remains to show that these 5 normal modes are linearly independent. Suppose there exists a superposition of these 5 modes that equals 0. By inspection, we see that normal mode D makes bars 2 and 6, 3 and 7, 4 and 8 not parallel, while normal mode E makes bars 1 and 5, 2 and 6, 4 and 8 not parallel. All other normal modes (A, B, and C) preserve the parallelism of all pairs of opposite bars. Thus, the amplitudes of D and E are 0. To see why the amplitudes of A, B, and C must also be zero, consider the following argument: Note that $\hat{\mathbf{z}} \times \mathbf{v}_1 = \mathbf{v}_3$ holds for both B and C, so thus it must hold for any linear combination of B and C. This means it needs to hold for A as well. However, we know $-\hat{\mathbf{z}} \times \mathbf{v}_1 = \mathbf{v}_3$ for A, which means the amplitude of A is 0. Since B and C are clearly not multiples of each other, we have that their amplitudes are 0. Therefore, all amplitudes are 0, which means A, B, C, D, and E are linearly independent.

Thus, we have 5 normal modes with 3 distinct natural oscillation frequencies, namely $\sqrt{\frac{24}{5-3\sqrt{2}}\frac{k}{ml^2}}$, $\sqrt{\frac{12}{5}\frac{k}{ml^2}}$, and $\sqrt{\frac{48+24\sqrt{2}}{5}\frac{k}{ml^2}}$. Since the ratios between these frequencies are irrational, there are no more natural oscillation frequencies of the system.

Note that 1 linearly independent oscillation mode corresponds to the first frequency (ω_A), 2 to the second (ω_B and ω_C), and 2 to the third (ω_D and ω_E).