Physics Cup 2021, Problem 3

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The total number of normal modes (including degenerate modes) is equal to the number of degrees of freedom in a mechanical system, so we first determine these. A general octagon is defined in a plane by its vertices, so we have 16 coordinates to start with. Now we apply the constraints that fix the distance between certain vertices to l. There are 8 of them, so we have $16-8=8$ independent coordinates remaining. Now there are 3 trivial independent solutions to the equations of motion that do not provide any oscillations: translational motion along the 2 planar axes with constant linear velocity, and rotation of the whole system with constant angular velocity. So we need to subtract them as well, so we finally get $8-3=5$ oscillation modes. So if we find (by clever guessing) 5 linearly independent oscillation modes along with their frequencies, the problem will be solved.

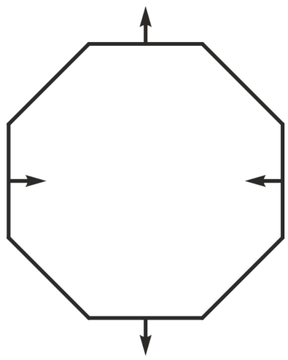

Fig. 1

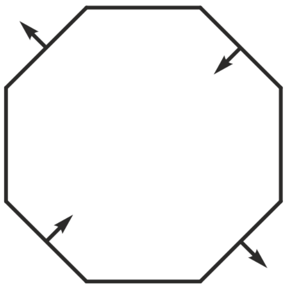

Fig. 2

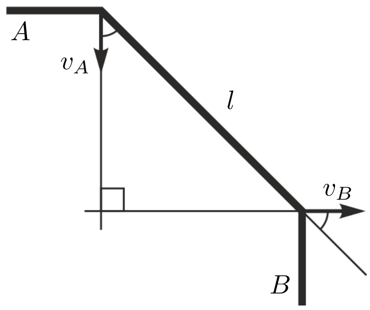

Fig. 3

The first oscillation mode can be seen easily: it is represented schematically in the fig. 1. It consists of two pairs of opposite sides oscillating with the same amplitude perpendicular to themselves without rotating. Kinematic constraints allow to relate the velocities of two perpendicular bars (as bars A and B in the fig. 3) at equilibrium (and also in a close vicinity of the equilibrium, since we consider only small oscillations). Projections of the velocities v_A and v_B on the diagonal bar (angled at 45°) must be equal, and since the angles between this bar and the velocities are 45° each, the velocities must also be equal themselves:

$$v_A = v_B = v.$$

This allows us to calculate the angular velocity of the diagonal bar:

$$\omega = \frac{|\mathbf{v}_A - \mathbf{v}_B|}{l} = \frac{v\sqrt{2}}{l}, \tag{1}$$

as well as its center-of-mass velocity:

$$v_c = \frac{|\mathbf{v}_A + \mathbf{v}_B|}{2} = \frac{v}{\sqrt{2}}. \tag{2}$$

Multiplying (1) by some small time Δt, and defining a first order linear displacement of the bar by $x = v\Delta t$, we get the rotation angle of the bar:

$$\Delta\varphi = \frac{x\sqrt{2}}{l}.$$

Now we write down the expression for the kinetic energy, which consists of 4 translational energies of the oscillating bars plus 4 energies of the diagonal bars, which in turn can be calculated using the König's theorem by substituting (1) and (2):

$$K = 4\cdot\frac{mv^2}{2} + 4\cdot\left(\frac{mv_c^2}{2} + \frac{ml^2}{12}\frac{\omega^2}{2}\right) = \frac{10}{3}mv^2.$$

Potential energy is the energy provided by the torque in the connectors. Similar to a spring's potential energy, this energy can be evaluated as

$$\Pi = 8\cdot\frac{k(\Delta\varphi)^2}{2} = \frac{8kx^2}{l^2}.$$

From the theory of harmonic oscillations of conservative systems follows that the square of the oscillation frequency is equal to the ratio of coefficients of x^2 in the potential energy to that of v^2 in the kinetic energy. Hence

$$\boxed{\Omega_{1,2}^2 = \frac{12}{5}\frac{k}{ml^2}.}$$

The subscript (1,2) indicates that there are actually two linearly independent modes associated with this frequency. The second one is exactly like the first one, except rotated 45 degrees (see fig. 2).

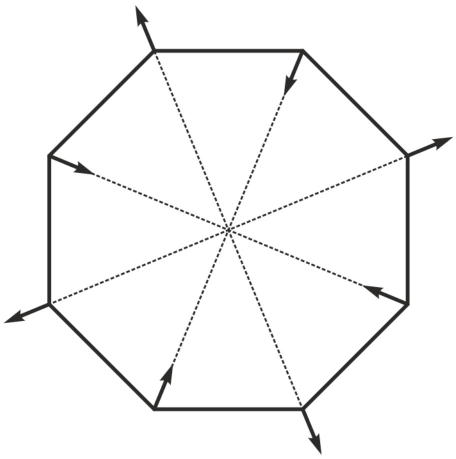

Fig. 4

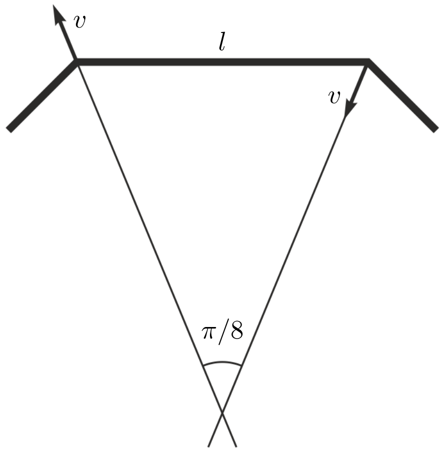

Fig. 5

The third mode can be illustrated in the fig. 4. The vertices of the octagon stay on their initial radial lines and oscillate in anti-phase. Due to the symmetry of the system the velocities of the vertices (and their displacements) are equal. We can, in the same way as for the first two modes, evaluate the required velocities and angles (see fig. 5 for reference):

$$v_c = v\sin\frac{\pi}{8} = v\sqrt{\frac{1-\sqrt{2}/2}{2}} = \frac{v\sqrt{2-\sqrt{2}}}{2},$$

$$\omega = \frac{2v\cos\frac{\pi}{8}}{l} = \frac{v\sqrt{2+\sqrt{2}}}{l},$$

$$\Delta\varphi = \frac{x\sqrt{2+\sqrt{2}}}{l}.$$

Similarly, the energies would be equal to

$$K = 8\cdot\left(\frac{mv_c^2}{2} + \frac{ml^2}{12}\frac{\omega^2}{2}\right) = \frac{8-2\sqrt{2}}{3}mv^2,$$

$$\Pi = 8\cdot\frac{k(2\Delta\varphi)^2}{2} = \frac{16(2+\sqrt{2})kx^2}{l^2}.$$

Thus

$$\boxed{\Omega_3^2 = \frac{24(2+\sqrt{2})}{4-\sqrt{2}}\frac{k}{ml^2}.}$$

In order to find the remaining frequency (frequencies, if there are 2), we combine in some way the two aforementioned motions. For example, we could try oscillations depicted in the fig. 6. The top and bottom bars move perpendicular to themselves without rotating, and right and left ones move in a similar fashion to the third mode. In order for the center of mass to stay at rest, the vertical velocities of the vertices belonging to the vertical bars need to equal in magnitude and opposite in direction to the velocities of the horizontal bars. That is, because the upward displacement of the 2 horizontal bars needs to be compensated by the downward displacement of the 2 vertical bars. Writing projections of the velocity components on the diagonal bars (see fig. 7), we, analogously to previous calculations, find the corresponding horizontal velocity to be $u = 2v$. From here we can easily find

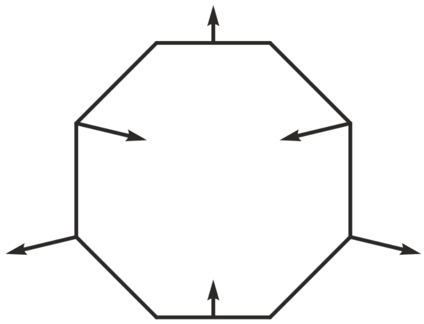

Fig. 6

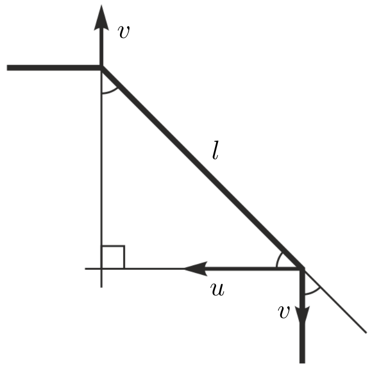

Fig. 7

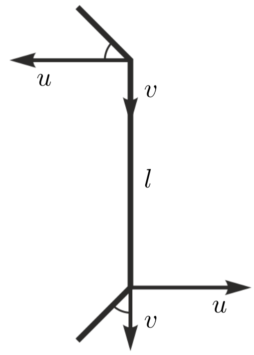

Fig. 8

the center-of-mass velocity of diagonal bars, as well as the angular velocity and rotation angle (using the same expression as in (1) and (2)):

$$v_c = v, \quad \omega = \frac{2\sqrt{2}\,v}{l}, \quad \Delta\varphi = \frac{2\sqrt{2}\,x}{l}.$$

For the vertical bars (fig. 8) we find

$$v_c' = v, \quad \omega' = \frac{4v}{l}, \quad \Delta\varphi' = \frac{4x}{l}.$$

The energies are equal to

$$K = 2 \cdot \frac{mv^2}{2} + 4 \cdot \left(\frac{mv_c^2}{2} + \frac{ml^2}{12}\frac{\omega^2}{2} \right) + 2 \cdot \left(\frac{mv_c'^2}{2} + \frac{ml^2}{12}\frac{\omega'^2}{2} \right) = \frac{20}{3}\,mv^2;$$

$$\Pi = 4 \cdot \frac{k(\Delta\varphi)^2}{2} + 4 \cdot \frac{k(\Delta\varphi + \Delta\varphi')^2}{2} = \frac{32(2+\sqrt{2})kx^2}{l^2}.$$

Finally we get the frequency

$$\boxed{\Omega_{4,5}^2 = \frac{24(2+\sqrt{2})}{5}\frac{k}{ml^2}.}$$

This frequency is different from what we have calculated earlier, which means that this is an entirely new mode, linearly independent of the previous ones. But that, in turn, means that a mode which is the same as the 4-th mode, but rotated 90 degrees, is also a new mode, since it has the same frequency as the 4-th mode and is linearly independent of it. So there are 2 modes associated with this last frequency, hence the subscript (4,5). This concludes the solution, because we found in total 5 linearly independent modes (see first paragraph).