Problem 3 Oscillating Octagon

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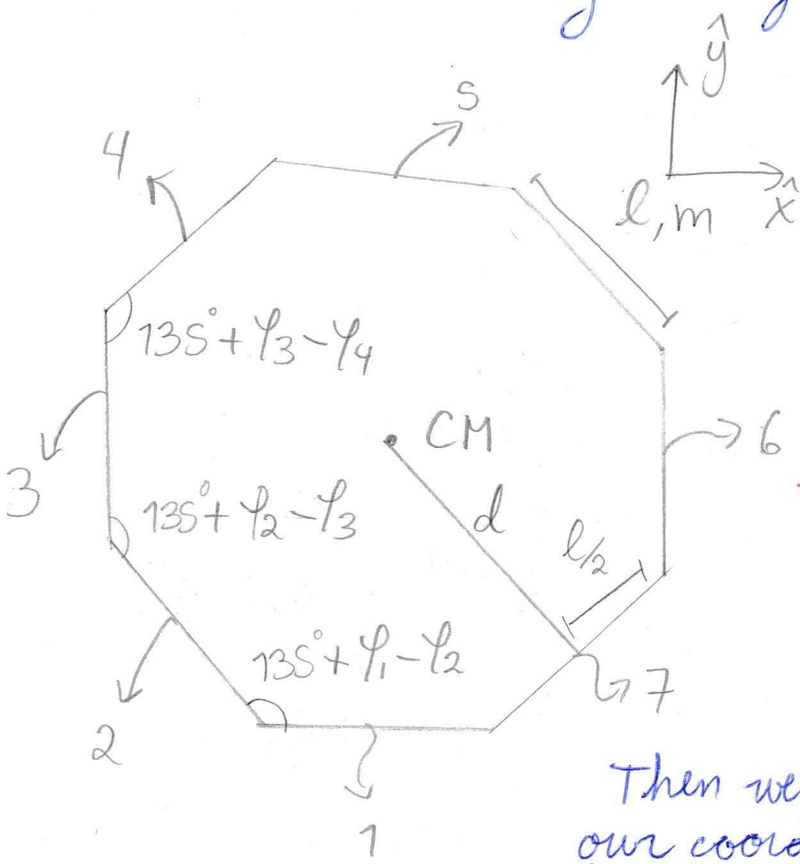


The first thing to solve the problem is to define our generalized coordinates and the origin point:

- We will choose the rotation angle of the bars $\varphi_1, \varphi_2 \dots \varphi_8$ as our generalized coordinates and the center of mass as our origin.

$$T = K\left(\varphi - \frac{3}{4}\pi\right)$$

$$\downarrow$$

$$135^\circ + \varphi_i - \varphi_j$$

Then we can use relations between our coordinates since the octagon is closed. Using complex numbers:

$$e^{i\varphi_1} + e^{i\left(\frac{\pi}{4}+\varphi_2\right)} + e^{i\left(\frac{\pi}{2}+\varphi_3\right)} + \dots e^{i\left(\frac{7}{4}\pi+\varphi_8\right)} = 0$$

But $\varphi_i \ll$ so

$$\varepsilon 1 \longrightarrow \quad \varphi_1 + \varphi_2 e^{i\pi/4} + \varphi_3 e^{i\pi/2} + \dots \varphi_8 e^{i\frac{7}{4}\pi} = 0$$

The next step is to find all possible vibration modes:

* There are the obvius ones where the whole octagon moves in x or y at constant speed, but they are eliminated when considering the CM as the origin. However, the eigenvectors would be (0,0,0,0,0,0,0,0)

* There is also the vibration mode where whole octagon rotates at constant speed, with eigenvector (1,1,1,1,1,1,1,1)

$$\Phi_0 = (1,1,1,1,1,1,1,1)$$

* There are three other oscillation modes, with eigenvectors:

✓ $\Phi_1 = (0, -1, 0, 1, 0, -1, 0, 1)$

✓ $\Phi_2 = (1, 0, -1, 0, 1, 0, -1, 0)$

✓ $\Phi_3 = (1, -1, 1, -1, 1, -1, 1, -1)$

As we can see on hint 3.

Remember that the eigenvectors must have propierties, for example:

$\Phi_i \cdot \Phi_j = 0$ and There are independents of each other.

Using these two properties we can find the two missing eigenvectors.

$\Phi_4 = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$

$\Phi_0 \cdot \Phi_4 = 0 \longrightarrow x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 0$ (1)

$\Phi_1 \cdot \Phi_4 = 0 \longrightarrow -x_2 + x_4 - x_6 + x_8 = 0$ (2)

$\Phi_2 \cdot \Phi_4 = 0 \longrightarrow x_1 - x_3 + x_5 - x_7 = 0$ (3)

$\Phi_3 \cdot \Phi_4 = 0 \longrightarrow x_1 + x_3 + x_5 + x_7 = x_2 + x_4 + x_6 + x_8$ (4)

Using (1) and (4)

$x_1 + x_3 + x_5 + x_7 = x_2 + x_4 + x_6 + x_8 = 0$ (5)

Using (5) and (3) Then

$x_1 = -x_5 \qquad x_2 = -x_6$

$x_3 = -x_7 \qquad x_4 = -x_8$

We can also use $\mathcal{E}_1$

$$\varphi_1 + \varphi_2 e^{i\pi/4} + \varphi_3 e^{i\frac{\pi}{2}} \cdots \varphi_8 e^{i\frac{7}{4}\pi} = 0$$

$$e^{i\varphi} = -e^{i(\pi+\varphi)}$$

$$2\left(\varphi_1 + \varphi_2 e^{i\pi/4} + \varphi_3 e^{i\frac{\pi}{2}} + \varphi_4 e^{i\frac{3}{4}\pi}\right) = 0$$

$$\varphi_1 + \varphi_2(\cos 45^\circ + i\sin 45^\circ) + \varphi_3 i + \varphi_4(-\cos 45^\circ + i\sin 45^\circ) = 0$$

$$\varphi_1 + (\varphi_2 - \varphi_4)\cos 45^\circ + i\left(\varphi_3 + (\varphi_2 + \varphi_4)\sin 45^\circ\right) = 0$$

$$\varphi_1 = -(\varphi_2 - \varphi_4)\cos 45^\circ \qquad \varphi_{i(t)} = \Phi\, \varphi_{(t)}$$

$$\varphi_3 = -(\varphi_2 + \varphi_4)\sin 45^\circ$$

$$X_1 = -(X_2 - X_4)\cos 45^\circ$$
$$X_3 = -(X_2 + X_4)\sin 45^\circ \Rightarrow \boxed{\begin{aligned} X_3 &= -X_1 - X_2\sqrt{2} \\ X_4 &= X_1\sqrt{2} + X_2 \end{aligned}}$$

$$X_2 = -X_4 - X_3\sqrt{2}$$
$$X_1 = X_3 + X_4\sqrt{2}$$

*

With those relations we can know Φ_4

$\Phi_4 = (0, 1, -\sqrt{2}, 1, 0, 1, \sqrt{2}, -1)$

And Φ_5 must be perpendicular to Φ_4 so

$\Phi_5 = (X_1, X_2, X_3, X_4 \cdots)$

$$X_2 - X_3\sqrt{2} + X_4 = 0$$
$$-X_4 - X_3\sqrt{2} - X_3\sqrt{2} + X_4 = 0$$
$$-2X_3\sqrt{2} = 0$$
$$X_3 = 0$$
$$X_1 = X_4\sqrt{2}, \quad X_2 = -X_4$$

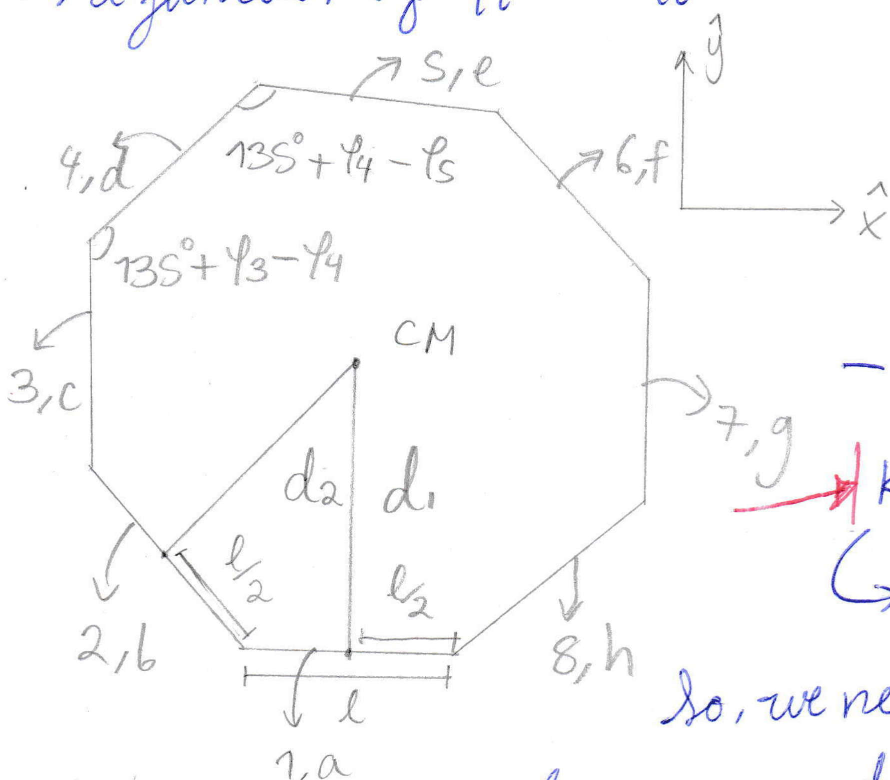
With this we know that

$\Phi_5 = (\sqrt{2}, -1, 0, 1, -\sqrt{2}, 1, 0, -1)$

And we have all the oscillation modes

$$\Phi_i \cdot \Phi_j = 0$$

The next step is to find the potential and kinetic energy as a function of φ_i and $\dot{\varphi}_i$.



- By the symmetry of our oscillation modes we can say that the CM will not move.

- Using König theorem

$$K_i = \frac{1}{2} m v_{cm_i}^2 + \frac{1}{2} I_{cm} \dot{\varphi}_i^2$$

↳ kinetic of each bar

So, we need to know v_{cm} for each bar

d_i is the distance between the CM of the octagon and the CM of i-bar

Using complex numbers:

$$d_1 + d_2 + d_3 + \cdots d_8 = 0$$

Because CM is the origin point, and all bars have the same mass

We can write d_i as a function of d_1 and φ_i

$$8d_1 + \frac{7}{2}a + 7\frac{b}{2} + \frac{6b}{2} + \frac{6c}{2} + \frac{5c}{2} + \cdots \frac{2g}{2} + \frac{h}{2} = 0$$

The octagon is closed so $a + b + c + \cdots g + h = 0$

Then

$$8d_1 + 3a + 6b + 5c + 4d + \cdots 2f + g = 0$$

$$\frac{8d_1}{l} + 3e^{i\varphi_1} + 6e^{i(\frac{1}{4}\pi + \varphi_2)} + 5e^{i(\frac{1}{2}\pi + \varphi_3)} + \cdots e^{i(\frac{3}{2}\pi + \varphi_7)} = 0$$

Remember $\varphi_i \ll$

$$\frac{8d_1}{l} + 3(1+i\varphi_1) + 6e^{\frac{1}{4}\pi}(1+i\varphi_2) + 5e^{\frac{1}{2}\pi}(1+i\varphi_3) + \cdots e^{\frac{3}{2}\pi}(1+i\varphi_7) = 0$$

$$\frac{8d_1}{l} + 3i(\varphi_1 - \varphi_5) + 4(e^{i\frac{3}{4}\pi} + e^{i\frac{1}{4}\pi} + e^{i\frac{1}{2}\pi}) + 4\varphi_4 i e^{\frac{3}{4}i\pi} + i e^{\frac{1}{4}i\pi}(6\varphi_2 - 2\varphi_6) + i e^{\frac{1}{2}i\pi}(5\varphi_3 - \varphi_7) = 0$$

$$\frac{8d_1}{l} + 3i(\varphi_1 - \varphi_5) + 4i(\sqrt{2}+1) + 4\varphi_4 i e^{\frac{3}{4}i\pi} + i e^{\frac{1}{4}i\pi}(6\varphi_2 - 2\varphi_6) + i e^{\frac{1}{2}i\pi}(5\varphi_3 - \varphi_7) = 0$$

$$i\frac{8d_1}{l} = 3(\varphi_1 - \varphi_5) + 4(\sqrt{2}+1) + 4\varphi_4 e^{\frac{3}{4}i\pi} + e^{\frac{1}{4}i\pi}(6\varphi_2 - 2\varphi_6) + e^{\frac{1}{2}i\pi}(5\varphi_3 - \varphi_7) \qquad \varepsilon 2$$

With this relation we can find d for each bar in each of the oscillation modes. Deriving we find v_{cm}

The expression of the potential energy is only:

$$U = \frac{1}{2}k\left(\varphi - \frac{3}{4}\pi\right)^2$$

$$U = \frac{1}{2}k(\varphi_i - \varphi_{i+1})^2$$

$$U_T = \frac{1}{2}k\sum_{i=1}^{8}(\varphi_i - \varphi_{i+1})^2 \qquad \varphi_9 = \varphi_1$$

Now only remains to obtain the kinetic and potential energy and it's average for each oscillation mode.

$* \quad \Phi_0 = (1,1,1,1,1,1,1,1)$ We can see that $w_0 = 0$. but let's check it

$$i\frac{8d}{l} = 4(\sqrt{2}+1) + 4\varphi\left[e^{\frac{3}{4}i\pi} + e^{\frac{1}{4}i\pi} + e^{\frac{1}{2}i\pi}\right]$$

$$\frac{8\dot{d}}{l} = 4\dot{\varphi}(\sqrt{2}+1)$$

$$\dot{d} = \frac{1}{2}(\sqrt{2}+1)\dot{\varphi} l$$

$$K = 8\left(\frac{1}{2}m\left(\frac{1}{2}(\sqrt{2}+1)\dot{\varphi} l\right)^2 + \frac{1}{2}I_{cm}\dot{\varphi}^2\right)$$

$$K = 8\dot{\varphi}^2\left(\frac{1}{2}m\left[\frac{1}{2}(\sqrt{2}+1)l\right]^2 + \frac{1}{2}I_{cm}\right)$$

$$\langle K\rangle = 8\varphi_0^2 w_0^2 \cdot \frac{1}{2}\left(\frac{1}{2}m\left(\frac{1}{2}(\sqrt{2}+1)l\right)^2 + \frac{1}{2}I_{cm}\right)$$

$$\langle K\rangle = \langle U\rangle \longrightarrow w_0 = 0 \ \frac{rad}{seg}$$

$$U = \frac{1}{2}k\Sigma(\varphi_i - \varphi_{i+1})^2$$

$$\varphi_i = \varphi_{i+1}$$

$$\Rightarrow U = 0$$

$$\langle U\rangle = 0$$

$$\varphi = \varphi_0\cos(wt+\phi)$$

$$\dot{\varphi} = -\varphi_0 w\sin(wt+\phi)$$

$$\langle\dot{\varphi}^2\rangle = \varphi_0^2 w^2\langle\sin^2(wt+\phi)\rangle$$

$$\langle\dot{\varphi}^2\rangle = \frac{1}{2}\varphi_0^2 w^2$$

$$\langle\varphi^2\rangle = \frac{1}{2}\varphi_0^2$$

$* \quad \Phi_1 = (0,-1,0,1,0,-1,0,1)$

$$i\frac{8d_1}{l} = 4(\sqrt{2}+1) + 4\varphi e^{\frac{3}{4}i\pi} - 4\varphi e^{\frac{1}{4}i\pi}$$

$$i\frac{8d_1}{l} = 4(\sqrt{2}+1) - 4\sqrt{2}\varphi$$

We are only interested in the magnitude, not the sign

$$* \quad \dot{d}_1 = \frac{\sqrt{2}}{2}\dot{\varphi} l = \dot{d}_3 = \dot{d}_5 = \dot{d}_7$$

$$I_{cm} = \frac{1}{12}ml^2$$

$$i\frac{8d_2}{l} = 4(\sqrt{2}+1) + 4\varphi i$$

$$* \quad \dot{d}_2 = \frac{1}{2}\dot{\varphi} l = \dot{d}_4 = \dot{d}_6 = \dot{d}_8$$

$$K = 4\left(\frac{1}{2}m\left[\left(\frac{\sqrt{2}}{2}\dot{\varphi} l\right)^2 + \left(\frac{1}{2}\dot{\varphi} l\right)^2\right] + \frac{1}{2}I_{cm}\dot{\varphi}^2\right)$$

$$K = 4\dot{\varphi}^2\left(\frac{1}{2}m\left[\frac{3}{4}l^2\right] + \frac{1}{2}I_{cm}\right)$$

$$\langle K\rangle = 2\varphi_1^2 w_1^2\left(\frac{5}{12}ml^2\right)$$

$$\langle K\rangle = \frac{5}{6}\varphi_1^2 w_1^2 ml^2$$

$$U = \frac{1}{2}k\Sigma(\varphi_i - \varphi_{i+1})^2 = \frac{1}{2}k(8\varphi^2) = 4k\varphi^2$$

$$\langle U\rangle = 2k\varphi_1^2$$

$$\langle K\rangle = \langle U\rangle$$

$$\frac{5}{6}ml^2\varphi_1^2 w_1^2 = 2k\varphi_1^2$$

$$w_1^2 = \frac{12}{5}\frac{k}{ml^2}$$

$$\boxed{w_1 = \sqrt{\frac{12}{5}\frac{k}{ml^2}}}$$

$*$ $\Phi_2 = (1, 0, -1, 0, 1, 0, -1, 0)$ We can expect that $w_2 = w_1$, but let's check it

$$i\frac{8d_1}{l} = 4(\sqrt{2}+1) - 4\varphi i$$

$$* \; \dot{d}_1 = \frac{1}{2}\dot{\varphi}l = \dot{d}_3 = \dot{d}_5 = \dot{d}_7$$

$$i\frac{8d_2}{l} = 4(\sqrt{2}+1) + 4\varphi e^{\frac{3}{4}i\pi} - 4\varphi e^{\frac{1}{4}i\pi}$$

$$i\frac{8d_2}{l} = 4(\sqrt{2}+1) - 4\sqrt{2}\varphi$$

$$* \; \dot{d}_2 = \frac{\sqrt{2}}{2}\dot{\varphi}l = \dot{d}_4 = \dot{d}_6 = \dot{d}_8$$

$$K = 4\left(\frac{1}{2}m\left[\left(\frac{\sqrt{2}}{2}\dot{\varphi}l\right)^2 + \left(\frac{1}{2}\dot{\varphi}l\right)^2\right] + \frac{1}{2}I_{cm}\dot{\varphi}^2\right)$$

$$K = 4\dot{\varphi}^2\left(\frac{1}{2}m\left(\frac{3}{4}l^2\right) + \frac{1}{2}I_{cm}\right)$$

$$\langle K\rangle = 2\varphi_2^2 w_2^2\left(\frac{5}{12}ml^2\right) = \frac{5}{6}w_2^2\varphi_2^2 ml^2$$

$$U = \frac{1}{2}k\Sigma(\varphi_i - \varphi_{i+1})^2$$

$$U = \frac{1}{2}k(8\varphi^2)$$

$$\langle U\rangle = 4k\langle\varphi^2\rangle$$

$$\langle U\rangle = 2k\varphi_2^2$$

$$\langle K\rangle = \langle U\rangle$$

$$\frac{5}{6}w_2^2\varphi_2^2 ml^2 = 2k\varphi_2^2$$

$$w_2^2 = \frac{12}{5}\frac{k}{ml^2}$$

$$\boxed{w_2 = \sqrt{\frac{12}{5}\frac{k}{ml^2}}}$$

* $\Phi_3 = (1,-1,1,-1,1,-1,1,-1)$

$$i\,\frac{8\dot{d}_1}{\ell} = 4(\sqrt{2}+1) - 4\varphi e^{\frac{3}{4}i\pi} - 4\varphi e^{\frac{1}{4}i\pi} + 4\varphi i$$

$$i\,\frac{8\dot{d}_1}{\ell} = 4(\sqrt{2}+1) - 4\varphi i(\sqrt{2}-1)$$

* $\dot{d}_1 = \frac{\sqrt{2}-1}{2}\dot{\varphi}\ell = \dot{d}_3 = \dot{d}_5 = \dot{d}_7$

$$i\,\frac{8\dot{d}_2}{\ell} = 4(\sqrt{2}+1) + 4\varphi e^{\frac{3}{4}i\pi} + 4\varphi e^{\frac{1}{4}i\pi} - 4i\varphi$$

$$i\,\frac{8\dot{d}_2}{\ell} = 4(\sqrt{2}+1) + 4i\varphi(\sqrt{2}-1)$$

* $\dot{d}_2 = \frac{\sqrt{2}-1}{2}\dot{\varphi}\ell = \dot{d}_4 = \dot{d}_6 = \dot{d}_8$

$$K = 8\left(\frac{1}{2}m\left(\frac{\sqrt{2}-1}{2}\dot{\varphi}\ell\right)^2 + \frac{1}{2}I_{cm}\dot{\varphi}^2\right)$$

$$K = 4\dot{\varphi}^2\left(m\left(\frac{3}{4}-\frac{\sqrt{2}}{2}\right)\ell^2 + \frac{1}{12}m\ell^2\right)$$

$$K = m\ell^2\dot{\varphi}^2\left(\frac{10}{3} - 2\sqrt{2}\right)$$

$$\langle K\rangle = \left(\frac{5}{3} - \sqrt{2}\right)m\ell^2\varphi_3^2\omega_3^2$$

$$U = \frac{1}{2}\kappa\sum(\varphi_i - \varphi_{i+1})^2$$

$$U = 16\kappa\varphi^2$$

$$\langle U\rangle = 8\kappa\varphi_3^2$$

$$\langle U\rangle = \langle K\rangle$$

$$\left(\frac{5}{3} - \sqrt{2}\right)m\ell^2\varphi_3^2\omega_3^2 = 8\kappa\varphi_3^2$$

$$\omega_3^2 = \frac{24}{5-3\sqrt{2}}\,\frac{\kappa}{m\ell^2}$$

$$\boxed{\omega_3 = \sqrt{\frac{24}{5-3\sqrt{2}}\,\frac{\kappa}{m\ell^2}}}$$

$$\Phi_4 = (0, 1, -\sqrt{2}, 1, 0, -1, \sqrt{2}, -1) \; *$$

$$i\,\frac{8d_1}{\ell} = 4(\sqrt{2}+1) + 4\varphi e^{\frac{3}{4}i\pi} + 8\varphi e^{\frac{1}{4}i\pi} - 6\sqrt{2}\,i$$

$$i\,\frac{8d_1}{\ell} = 4(\sqrt{2}+1) + 4\varphi\left(\frac{\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i\right) - 6\sqrt{2}\,i$$

$$i\,\frac{8d_1}{\ell} = 4(\sqrt{2}+1) + 2\sqrt{2}\,\varphi$$

$$* \; \dot{d}_1 = \frac{\sqrt{2}}{4}\dot{\varphi}\ell$$

$$i\,\frac{8d_2}{\ell} = 6\varphi + 4(\sqrt{2}+1) - 8\sqrt{2}\,\varphi e^{\frac{1}{4}i\pi} + 6\varphi i$$

$$i\,\frac{8d_2}{\ell} = 6\varphi + 4(\sqrt{2}+1) - 8\varphi(1+i) + 6\varphi i$$

$$i\,\frac{8d_2}{\ell} = -2\varphi(1+i) + 4(\sqrt{2}+1)$$

$$8\dot{d}_2 = (2\dot{\varphi} i - 2\dot{\varphi} i)\ell$$

$$* \; \dot{d}_2 = \frac{\sqrt{2}}{4}\dot{\varphi}\ell$$

$$i\,\frac{8d_3}{\ell} = -6\sqrt{2}\,\varphi + 4(\sqrt{2}+1) - 4\varphi e^{\frac{3}{4}i\pi} + 8\varphi e^{\frac{1}{4}i\pi}$$

$$i\,\frac{8d_3}{\ell} = -6\sqrt{2}\,\varphi + 4(\sqrt{2}+1) + 4\varphi\left(\frac{3\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$$

$$i\,\frac{8d_3}{\ell} = 4(\sqrt{2}+1) + 2\sqrt{2}\,\varphi i$$

$$* \; \dot{d}_3 = \frac{\sqrt{2}}{4}\dot{\varphi}\ell$$

$$i\frac{8d_4}{l} = 6\varphi + 4(\sqrt{2}+1) + 4\sqrt{2}\,\varphi e^{\frac{3}{4}i\pi} - 6\varphi i$$

$$i\frac{8d_4}{l} = 6\varphi + 4(\sqrt{2}+1) + 4\varphi(i-1) - 6\varphi i$$

$$i\frac{8d_4}{l} = 2\varphi - 2\varphi i + 4(\sqrt{2}+1)$$

$$\dot{d_4} = \frac{1}{4}l(\dot{\varphi} - \dot{\varphi}i)$$

By symmetry

$$\dot{d_5} = \dot{d_6} = \dot{d_7} = \dot{d_8} = \dot{d_1}$$

$$* \; \dot{d_4} = \frac{\sqrt{2}}{4}\dot{\varphi}l$$

$$K = 8\left(\frac{1}{2}m\left(\frac{\sqrt{2}}{4}\dot{\varphi}l\right)^2\right) + I_{cm}(\dot{\varphi}^2 + 2\dot{\varphi}^2 + \dot{\varphi}^2)$$

$$K = 4\dot{\varphi}^2\left(\frac{1}{8}ml^2 + I_{cm}\right)$$

$$\langle K\rangle = 2\varphi_4^2 w_4^2\left(\frac{5}{24}ml^2\right) = \frac{5}{12}\varphi_4^2 ml^2 w_4^2$$

$$U = \frac{1}{2}k\sum(\varphi_i - \varphi_{i+1})^2$$

$$U = \frac{1}{2}k\left(4\varphi^2(1+(\sqrt{2}+1)^2)\right)$$

$$\langle U\rangle = k\varphi_4^2(1+(\sqrt{2}+1)^2) = k\varphi_4^2(4+2\sqrt{2})$$

$$\langle U\rangle = 2k\varphi_4^2(2+\sqrt{2})$$

$$\langle K\rangle = \langle U\rangle$$

$$\frac{5}{12}\varphi_4^2 ml^2 w_4^2 = 2k\varphi_4^2(2+\sqrt{2})$$

$$w_4^2 = \frac{24(2+\sqrt{2})}{5}\frac{k}{ml^2}$$

$$\rightarrow \; w_4 = \sqrt{\frac{24(2+\sqrt{2})}{5}\frac{k}{ml^2}}$$

$\Phi_5 = (\sqrt{2}, -1, 0, 1, -\sqrt{2}, 1, 0, -1)$ *

$$i\frac{8\dot{d}_1}{\ell} = 6\sqrt{2}\varphi + 4(\sqrt{2}+1) + 4\varphi e^{\frac{3}{4}i\pi} - 8\varphi e^{\frac{1}{4}i\pi}$$

$$i\frac{8\dot{d}_1}{\ell} = 6\sqrt{2}\varphi + 4(\sqrt{2}+1) + 4\varphi\left(-\frac{3\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$$

$$i\frac{8\dot{d}_1}{\ell} = 4(\sqrt{2}+1) - 2\sqrt{2}\varphi i$$

$$* \ \dot{d}_1 = \frac{\sqrt{2}}{4}\dot{\varphi}\ell$$

$$i\frac{8\dot{d}_2}{\ell} = -6\varphi + 4(\sqrt{2}+1) - 4\sqrt{2}e^{\frac{3}{4}i\pi} + 6\varphi i$$

$$i\frac{8\dot{d}_2}{\ell} = -6\varphi + 4(\sqrt{2}+1) - 4\sqrt{2}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\varphi + 6\varphi i$$

$$i\frac{8\dot{d}_2}{\ell} = -6\varphi + 4(\sqrt{2}+1) + 4\varphi - 4\varphi i + 6\varphi i$$

$$i\frac{8\dot{d}_2}{\ell} = -2\varphi + 2\varphi i + 4(\sqrt{2}+1)$$

$$\dot{d}_2 = \frac{1}{8}\ell(-2\dot{\varphi} + 2\dot{\varphi} i)$$

$$* \ \dot{d}_2 = \frac{\sqrt{2}}{4}\dot{\varphi}\ell$$

$$i\frac{8\dot{d}_3}{\ell} = 4(\sqrt{2}+1) + 4\varphi e^{\frac{3}{4}i\pi} + 8\varphi e^{\frac{1}{4}i\pi} - 6\sqrt{2}\varphi i$$

$$i\frac{8\dot{d}_3}{\ell} = 4(\sqrt{2}+1) + 4\varphi\left(\frac{\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i\right) - 6\sqrt{2}\varphi i$$

$$i\frac{8\dot{d}_3}{\ell} = 4(\sqrt{2}+1) + 2\sqrt{2}\varphi$$

$$* \ \dot{d}_3 = \frac{\sqrt{2}}{4}\dot{\varphi}\ell$$

$$i\frac{8d_4}{l} = 6\varphi + 4(\sqrt{2}+1) - 8\sqrt{2}\,e^{\frac{1}{4}i\pi} + 6\varphi i$$

$$i\frac{8d_4}{l} = 6\varphi + 4(\sqrt{2}+1) - 8\sqrt{2}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)\varphi + 6\varphi i$$

$$i\frac{8d_4}{l} = 6\varphi + 4(\sqrt{2}+1) - 4\varphi i - 4\varphi + 6\varphi i$$

$$i\frac{8d_4}{l} = 4(\sqrt{2}+1) + 2\varphi + 2\varphi i$$

$$\dot{d}_4 = \frac{1}{8}l(2\dot{\varphi} + 2\dot{\varphi}i)$$

By symmetry

$$\dot{d}_5 = \dot{d}_6 = \dot{d}_7 = \dot{d}_8 = \dot{d}_1$$

$$* \ \dot{d}_4 = \frac{\sqrt{2}}{4}\dot{\varphi}l$$

$$K = 8\left(\frac{1}{2}m\left(\frac{\sqrt{2}}{4}\dot{\varphi}l\right)^2\right) + I_{cm}\left(\dot{\varphi}^2 + \dot{\varphi}^2 + 2\dot{\varphi}^2\right))$$

$$K = 4\left(m\left(\frac{\sqrt{2}}{4}\dot{\varphi}l\right)^2 + I_{cm}\dot{\varphi}^2\right)$$

$$K = 4\dot{\varphi}^2\left(\frac{1}{8}ml^2 + I_{cm}\right)$$

$$\langle K\rangle = 2\varphi_s^2 w_s^2\left(\frac{5}{24}ml^2\right) = \frac{5}{12}ml^2\varphi_s^2 w_s^2$$

$$U = \frac{1}{2}k\Sigma(\varphi_i - \varphi_{i+1})^2$$

$$\langle K\rangle = \langle U\rangle$$

$$U = \frac{1}{2}k\left(4\varphi^2\left(1 + (\sqrt{2}+1)^2\right)\right)$$

$$\frac{5}{12}ml^2\varphi_s^2 w_s^2 = 2k\varphi_s^2(2+\sqrt{2})$$

$$\langle U\rangle = k\varphi_s^2(4 + 2\sqrt{2})$$

$$w_s^2 = \frac{24}{5}\frac{k}{ml^2}(2+\sqrt{2})$$

$$\langle U\rangle = 2k\varphi_s^2(2+\sqrt{2})$$

$$\boxed{w_s = \sqrt{\frac{24}{5}(2+\sqrt{2})\frac{k}{ml^2}}}$$

To end the solution, we have four non-trivial natural oscillation frequencies. There are:

* $w_0 = 0 \, \frac{rad}{seg}$ $\longrightarrow$ with eigenvectors (oscillation modes)

$(1,1,1,1,1,1,1,1)$
$(0,0,0,0,0,0,0,0)$ Eliminated when considering CM as origin

* $w_1 = \sqrt{\frac{12}{5}\frac{k}{ml^2}}$ $\longrightarrow$ $(0,-1,0,1,0,-1,0,1)$, $(1,0,-1,0,1,0,-1,0)$

* $w_2 = \sqrt{\frac{24}{5-3\sqrt{2}}\frac{k}{ml^2}}$ $\longrightarrow$ $(1,-1,1,-1,1,-1,1,-1)$

* $w_3 = \sqrt{\frac{24}{5}(2+\sqrt{2})\frac{k}{ml^2}}$ $\longrightarrow$ $(0,1,-\sqrt{2},1,0,-1,\sqrt{2},-1)$, $(\sqrt{2},-1,0,1,-\sqrt{2},1,0,-1)$

$\varphi_i = \Phi \, \varphi_0 \sin(wt + \phi)$

Φ: Corresponding eigenvector

(w_0 could be considered as a trivial frequency)

Φ differents for differents frequencies are orthogonal and linearly independents.