Physics Cup 2021 – Problem 3

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Problem

Find all non-trivial natural oscillation frequencies for a regular octagon made from eight homogeneous bars of mass m and length l. While the bars are rigid, the connectors connecting two neighboring bars are such that the angle φ between the bars can be changed without any friction, but a returning torque $\tau = k\left(\varphi - \frac{3}{4}\pi\right)$ will appear at the joint as soon as the angle departs from its equilibrium value $\frac{3}{4}\pi$. Indicate how many linearly independent oscillation modes correspond to each of these frequencies. Consider only planar oscillation modes, i.e. modes by which the bars move only in the plane of the octagon.

Solution

Continuing from Hint 3, we need to find two more vectors $\overrightarrow{\Phi}_4$ and $\overrightarrow{\Phi}_5$ that are orthogonal to $\overrightarrow{\Phi}_0$, $\overrightarrow{\Phi}_1$, $\overrightarrow{\Phi}_2$ and $\overrightarrow{\Phi}_3$, and also satisfy the constraint equation $\varphi_1 + e^{\frac{i\pi}{4}}\varphi_2 + \cdots + e^{i\frac{7\pi}{4}}\varphi_8 = 0$.

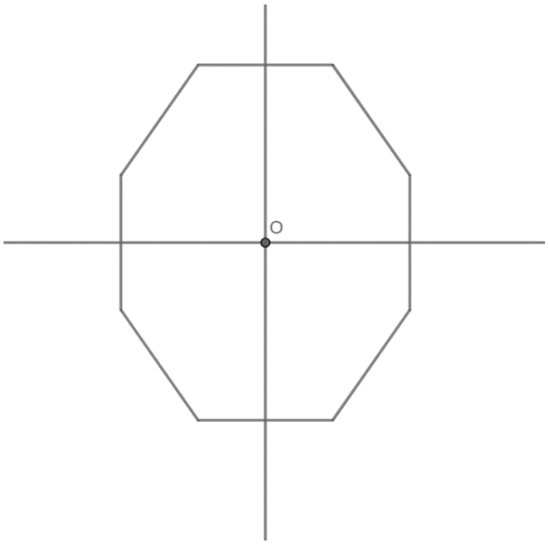
This is satisfied by the vectors

$$\overrightarrow{\Phi}_4 = \left(1, -\sqrt{2}, 1, 0, -1, \sqrt{2}, -1, 0\right)$$
$$\overrightarrow{\Phi}_5 = \left(-\sqrt{2}, 1, 0, -1, \sqrt{2}, -1, 0, 1\right)$$

Other rotations of the two vectors can be expressed as linear combinations of the two.
which can be easily checked to be orthogonal to $\overrightarrow{\Phi}_0 = (1,1,1,1,1,1,1,1)$, $\overrightarrow{\Phi}_1 = (1,0,-1,0,1,0,-1,0)$, $\overrightarrow{\Phi}_2 = (0,-1,0,1,0,-1,0,1)$.

Now we calculate the angular frequencies associated with each of the vectors.

For the cases of $\overrightarrow{\Phi}_1$ and $\overrightarrow{\Phi}_2$:



Suppose that the diagonal rods are angled $45^\circ + \theta$ to the horizontal. These rods undergo rotational motion and translational motion, while the vertical and the horizontal rods only undergo translational motion.

First, we calculate the coordinates of the midpoints of the rods.

The midpoint of the top horizontal rod has coordinates $\left(\frac{L}{2}, \frac{L}{2} + L\sin\left(\frac{\pi}{4} + \theta\right)\right)$.
The midpoint of the right vertical rod has coordinates $\left(\frac{L}{2} + L\cos\left(\frac{\pi}{4} + \theta\right), 0\right)$.
The midpoint of the right diagonal rod has coordinates $\left(\frac{L}{2} + \frac{L}{2}\cos\left(\frac{\pi}{4} + \theta\right), \frac{L}{2} + \frac{L}{2}\sin\left(\frac{\pi}{4} + \theta\right)\right)$.
We differentiate the three coordinates and find the speeds of the rods, ending up with

$$L\cos\left(\frac{\pi}{4} + \theta\right)\dot{\theta} \approx \frac{L}{\sqrt{2}}\dot{\theta} \text{ (horizontal)}$$

$$L\sin\left(\frac{\pi}{4} + \theta\right)\dot{\theta} \approx \frac{L}{\sqrt{2}}\dot{\theta} \text{ (vertical)}$$

$$\frac{L}{2}\dot{\theta} \text{ (diagonal)}$$

Finding the total kinetic energy of the rods, we have for the diagonal rods,

$$4 \cdot \left(\frac{1}{2}m\left(\frac{L\dot{\theta}}{2}\right)^2 + \frac{1}{2} \cdot \frac{1}{12}mL^2\dot{\theta}^2\right) = \frac{2}{3}mL^2\dot{\theta}^2$$

For the horizontal and vertical rods,

$$4 \cdot \frac{1}{2}m\left(\frac{L}{\sqrt{2}}\right)^2 \dot{\theta}^2 = mL^2\dot{\theta}^2$$

Hence,

$$KE_{tot} = \frac{5}{3}mL^2\dot{\theta}^2$$

The potential energy at each of the hinges is $\frac{1}{2}k\theta^2$ as the hinge is rotated θ from equilibrium. The total potential energy is thus $4k\theta^2$.

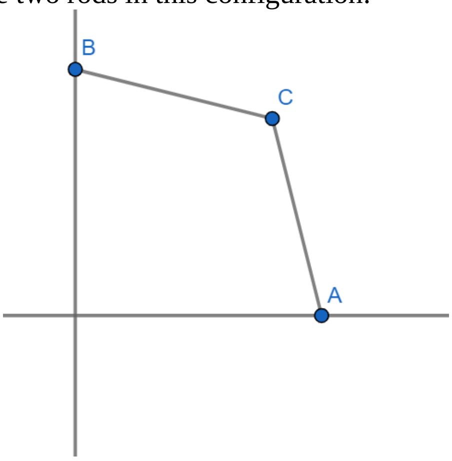
Writing the conservation of energy equation,

$$\frac{5}{3}mL^2\dot{\theta}^2 + 4k\theta^2 = \text{const}$$

This means that the angular frequency is

$$\omega = \sqrt{\frac{4k}{\frac{5}{3}mL^2}} = \sqrt{\boldsymbol{\frac{12k}{5mL^2}}}$$

For the case Φ_3, we consider the two rods in this configuration:



The points A and B slide along the axes such that AB is always $45°$ from the horizontal, and C is a point on the perpendicular bisector of AB.

Suppose that line BC makes an angle $\frac{3\pi}{8} - \theta$ from the y-axis. Then the diagonal AB has length $2L\cos\left(\frac{\pi}{8} - \theta\right)$ and the distance from the origin to B is $\sqrt{2}L\cos\left(\frac{\pi}{8} - \theta\right)$.

The midpoint of rod BC is thus

$$\left(\frac{L}{2}\sin\left(\frac{3\pi}{8}-\theta\right), \sqrt{2}L\cos\left(\frac{\pi}{8}-\theta\right)-\frac{L}{2}\cos\left(\frac{3\pi}{8}-\theta\right)\right)$$

Differentiating with respect to time, we get

$$\left(-\frac{L}{2}\cos\left(\frac{3\pi}{8}-\theta\right)\dot{\theta}, \sqrt{2}L\sin\left(\frac{\pi}{8}-\theta\right)-\frac{L}{2}\sin\left(\frac{3\pi}{8}-\theta\right)\right)\dot{\theta}$$

The square of the speed of the midpoint is thus

$$\left(\left(\frac{L}{2}\cos\left(\frac{3\pi}{8}-\theta\right)\right)^2+\left(\sqrt{2}L\sin\left(\frac{\pi}{8}-\theta\right)-\frac{L}{2}\sin\left(\frac{3\pi}{8}-\theta\right)\right)^2\right)\dot{\theta}^2$$
$$=\left(\frac{L^2}{4}+2L^2\sin^2\left(\frac{\pi}{8}-\theta\right)-\sqrt{2}L^2\sin\left(\frac{\pi}{8}-\theta\right)\sin\left(\frac{3\pi}{8}-\theta\right)\right)\dot{\theta}^2$$

The kinetic energy of one rod is thus

$$\frac{1}{2}\cdot\frac{1}{12}mL^2\dot{\theta}^2+\frac{1}{2}\cdot mv_{mid}^2\approx\left(\frac{1}{24}+\frac{1}{8}+\sin^2\frac{\pi}{8}-\frac{\sqrt{2}}{2}\sin\frac{\pi}{8}\sin\frac{3\pi}{8}\right)mL^2\dot{\theta}^2$$
$$=\left(\frac{5}{12}-\frac{1}{2\sqrt{2}}\right)mL^2\dot{\theta}^2$$

The total kinetic energy is $8\cdot\left(\frac{5}{12}-\frac{1}{2\sqrt{2}}\right)mL^2\dot{\theta}^2$.

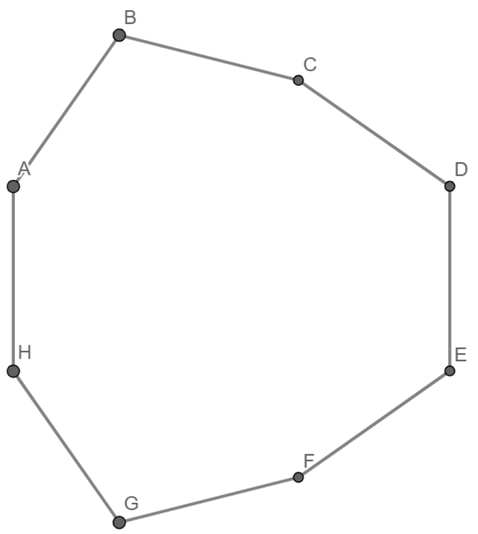
The total potential energy is $8\cdot\frac{k(2\theta)^2}{2}$.

Writing the conservation equation,

$$\left(\frac{5}{12}-\frac{1}{2\sqrt{2}}\right)mL^2\dot{\theta}^2+2k\theta^2=\text{const}$$

$$\omega=\sqrt{\frac{2k}{\left(\frac{5}{12}-\frac{1}{2\sqrt{2}}\right)mL^2}}=\sqrt{\boldsymbol{\frac{24k}{(5-3\sqrt{2})mL^2}}}$$

Lastly, we look at the case Φ_4.



Suppose AB is angled $\frac{\pi}{4}+\theta$ from the horizontal. First, we claim that AD is always horizontal. To see why, note that

$$\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}$$
$$=\left(L\cos\left(\frac{\pi}{4}+\theta\right), L\sin\left(\frac{\pi}{4}+\theta\right)\right)+\left(L\cos(\sqrt{2}\theta), -L\sin(\sqrt{2}\theta)\right)$$
$$+\left(L\cos\left(\frac{\pi}{4}-\theta\right), -L\sin\left(\frac{\pi}{4}-\theta\right)\right)$$

The y-component is equal to

$$L\left(\sin\left(\frac{\pi}{4}+\theta\right)-\sin(\sqrt{2}\theta)-\sin\left(\frac{\pi}{4}-\theta\right)\right)=L\left(2\cos\frac{\pi}{4}\sin\theta-\sin(\sqrt{2}\theta)\right)=L\cdot O(\theta^2)\approx 0$$

So we can assume that the vertical rods only move left and right.

The x-component can be approximated to be $2L\cos\frac{\pi}{4}+L+L\cdot O(\theta^2)$ which can be taken as a constant. Thus the distance between the vertical rods also does not change.

Now we go into the reference frame of the vertical rods and find the velocities of the three other rods on top. Suppose that A has coordinates $(0,0)$.

The midpoint of the rod AB has coordinates

$$\left(\frac{L}{2}\cos(\frac{\pi}{4}+\theta),\frac{L}{2}\sin\left(\frac{\pi}{4}+\theta\right)\right)$$

It has velocity

$$\left(-\frac{L}{2}\sin\left(\frac{\pi}{4}+\theta\right),\frac{L}{2}\cos\left(\frac{\pi}{4}+\theta\right)\right)\dot{\theta}\approx\left(-\frac{L}{2\sqrt{2}},\frac{L}{2\sqrt{2}}\right)\dot{\theta}$$

The midpoint of the rod BC has coordinates

$$\left(L\cos\left(\frac{\pi}{4}+\theta\right)+\frac{L}{2}\cos(\sqrt{2}\theta),L\sin\left(\frac{\pi}{4}+\theta\right)-\frac{L}{2}\sin(\sqrt{2}\theta)\right)$$

It has velocity

$$\left(-L\sin\left(\frac{\pi}{4}+\theta\right)-\frac{L}{\sqrt{2}}\sin(\sqrt{2}\theta),L\cos\left(\frac{\pi}{4}+\theta\right)-\frac{L}{\sqrt{2}}\cos(\sqrt{2}\theta)\right)\dot{\theta}\approx\left(-\frac{L}{\sqrt{2}},0\right)\dot{\theta}$$

The midpoint of the rod CD has coordinates

$$\left(L\cos\left(\frac{\pi}{4}+\theta\right)+L\cos(\sqrt{2}\theta)+\frac{L}{2}\cos\left(\frac{\pi}{4}-\theta\right),L\sin\left(\frac{\pi}{4}+\theta\right)-L\sin(\sqrt{2}\theta)-\frac{L}{2}\sin\left(\frac{\pi}{4}-\theta\right)\right)$$

It has velocity

$$\left(-L\sin\left(\frac{\pi}{4}+\theta\right)-\sqrt{2}L\sin(\sqrt{2}\theta)+\frac{L}{2}\sin\left(\frac{\pi}{4}-\theta\right),L\cos\left(\frac{\pi}{4}+\theta\right)-\sqrt{2}L\cos(\sqrt{2}\theta)\right.$$
$$\left.+\frac{L}{2}\cos\left(\frac{\pi}{4}-\theta\right)\right)\dot{\theta}\approx\left(-\frac{L}{2\sqrt{2}},-\frac{L}{2\sqrt{2}}\right)\dot{\theta}$$

Now suppose that the rods AH and DE move right with velocity v in the lab frame.

Using the fact that the CM does not move,

$$\left(-\frac{L\dot{\theta}}{2\sqrt{2}}+v\right)+\left(-\frac{L\dot{\theta}}{\sqrt{2}}+v\right)+\left(-\frac{L\dot{\theta}}{2\sqrt{2}}+v\right)+v=0$$

Hence,

$$v=\frac{1}{2\sqrt{2}}L\dot{\theta}$$

Meaning that the midpoints of rods AB and CD do not move horizontally, and rod BC moves to the left at a speed $\frac{1}{2\sqrt{2}}L\dot{\theta}$.

Now we compute the kinetic energies.

For the rods AB and CD, their kinetic energy is $\frac{1}{2}\cdot m\left(\frac{L}{2\sqrt{2}}\dot{\theta}\right)^2+\frac{1}{2}\cdot\frac{1}{12}mL^2\dot{\theta}^2=\frac{5}{48}mL^2\dot{\theta}^2$ (vertical + rotation).

For the rod BC, its kinetic energy is $\frac{1}{2}\cdot m\left(\frac{1}{2\sqrt{2}}L\dot{\theta}\right)^2+\frac{1}{2}\cdot\frac{1}{12}mL^2\left(\sqrt{2}\dot{\theta}\right)^2=\frac{7}{48}mL^2\dot{\theta}^2$ (horizontal + rotation).

For the vertical rods, their kinetic energy is $\frac{1}{2}\cdot m\left(\frac{1}{2\sqrt{2}}L\dot{\theta}\right)^2=\frac{1}{16}mL^2\dot{\theta}^2$ (horizontal).

This means that the total kinetic energy is $\left(4\cdot\frac{5}{48}+2\cdot\frac{1}{16}+2\cdot\frac{7}{48}\right)mL^2\dot{\theta}^2=\frac{5}{6}mL^2\dot{\theta}^2$.

The potential energy at the pivots at the vertical rods is $\frac{1}{2}k\theta^2$, while the potential energy at the pivots at the horizontal rods is $\frac{1}{2}k\left(\left(\sqrt{2}+1\right)\theta\right)^2$.

The total potential energy is

$$2k\theta^2 + 2k\left(\sqrt{2}+1\right)^2\theta^2 = 4\left(2+\sqrt{2}\right)k\theta^2$$

This means that

$$\frac{5}{6}mL^2\dot{\theta}^2 + 4\left(2+\sqrt{2}\right)k\theta^2 = \text{const}$$

$$\omega = \sqrt{\boldsymbol{\frac{24\left(2+\sqrt{2}\right)k}{5mL^2}}}$$