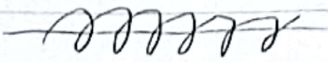
from $B >> \frac{m k_B T}{he}$



"radius" of spiral very small, only need to consider motion in z-direction

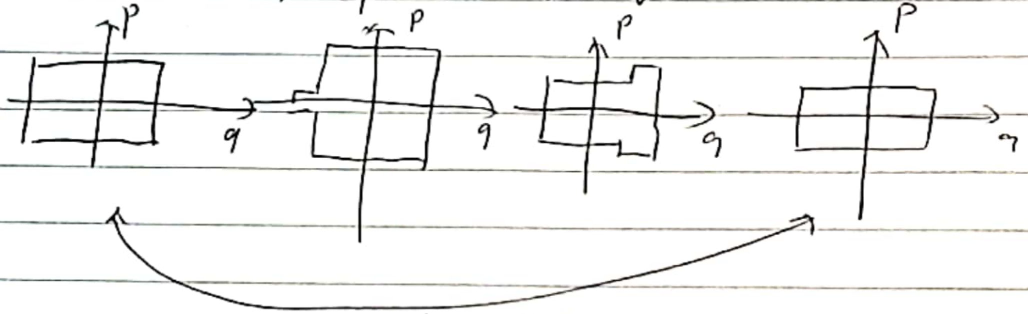
- for $\infty > \frac{1}{2} m V_{z_0}^2 > 100 k_B T$,

  the electron can always pass through the barrier, the "period" of the back and forth motion of electron is very small compared to motion of barrier ⇒ adiabatic invariant of $\int p\,dq$ where p is momentum in z, q is z coordinate.

  By principle of adiabatic invariance, $\oint p\,dq$ is constant.

  since initial and final $\Delta q = L$ unchanged

  then momentum, speed unchanged



- for $\frac{1}{2} m V_{z_0}^2 < 100 k_B T$

  the electron cannot pass through barrier initially. it is adiabatically compressed by the moving barrier, each time its speed increments by $2u$, until it has enough KE

  let $\frac{1}{2} m V_c^2 = 100 k_B T$

  let $V_{z,before}$ be velocity when it first cross barrier and $V_{z,after}$ be that after.

consider in the rest frame w.r.t. barrier

$$\frac{1}{2} m (V_{z,before} + u)^2 - \frac{1}{2} m V_c^2 = \frac{1}{2} m (V_{after} + u)^2$$

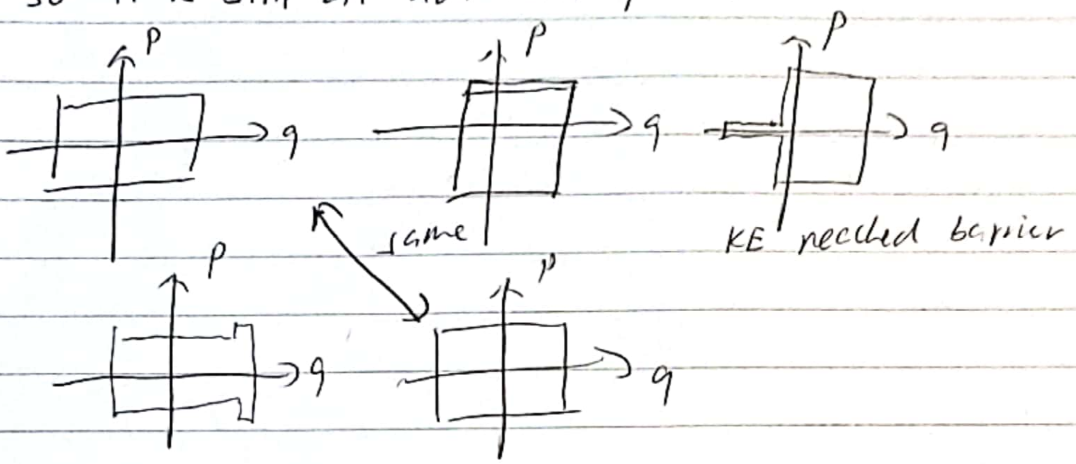
$$V_{after} + u = \sqrt{(V_{z,before} + u + V_c)(V_{z,before} + u - V_c)}$$

$$V_{z,before} + u - V_c \sim O(u)$$ in most case due to $2u$-increment

$$V_{after} \sim O\left(\sqrt{\frac{V_c}{u}} \cdot u\right)$$

$$\sim O\left(\sqrt{\frac{\sqrt{k_B T/m}}{u}} \cdot u\right) \quad >> u$$

so it is still an adiabatic process



by constant area argument final speed still equals initial speed.

Speed distribution is mostly identical (apart from a very tiny portion of electrons have almost zero speed after barrier cannot reach barrier again)

$\therefore P_{final} = P_0$ (in actual should be a little bit smaller)

A more detailed derivation from differentials.

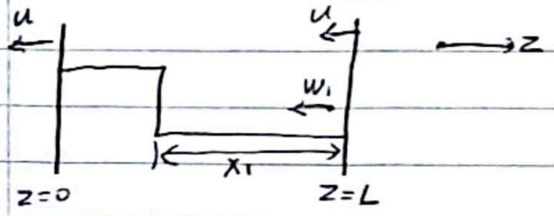
Consider the whole process in the reference frame co-moving with the $\frac{U_0}{e}H(ut-z)$ potential barrier. Because in this frame no external work done on barrier since barrier is static. (Note that in lab frame, in order to keep the barrier moving at speed u, external work done of Impulse $\cdot\, u$ for each collision, so EPE + KE of electron not conserved in original lab frame, but is conserved in this frame). (KE + EPE conserved in co-moving frame of barrier)

Note that the E and B fields in this frame is:

$$\vec{E}_z' = \vec{E}_{z\,lab}$$
$$\vec{B}_z' = \vec{B}_{z\,lab}$$
$$\vec{E}_\perp' = \gamma(\vec{E}_{\perp\,lab} + \vec{u}\times\vec{B}_{lab})$$
$$\vec{B}_\perp' = \gamma(\vec{B}_{\perp\,lab} - \frac{1}{c^2}\vec{u}\times\vec{E}_{lab})$$

Since we are only interested in z-direction and we assume $|\gamma - 1| \ll 1$ since u is much smaller than speed of light.

Phase 1: when the electron is trapped within the potential barrier



let w_0 be its initial speed $= V_{z0} - u$

let w_i be its speed after ith collision with right wall

let x_i be the distance between potential barrier and wall after ith collision.

$$W_1 = V_{z0} + u = (V_{z0} - u) + 2u \qquad x_1 \approx L$$
$$W_2 = V_{z0} + 3u \qquad x_2 = x_1 - \frac{2ux_1}{w_1+u} = \frac{w_1 - u}{w_1 + u}x_1$$
$$W_3 = V_{z0} + 5n \qquad x_3 = \frac{W_2 - u}{W_2 + u}x_2$$
$$\vdots \qquad \vdots$$
$$W_n = V_{z0} + (2n-1)u \qquad x_n = \frac{W_{n-1} - u}{W_{n-1} + u}x_{n-1}$$

$$\therefore \quad \Delta W = 2u$$

$$\Delta X = X_n - X_{n-1} = \frac{-2u}{W_{n-1}+u} X_{n-1} = \frac{-2u}{W} X \quad \text{since } u << W$$

$$\therefore \quad \frac{dW}{dX} \approx \frac{\Delta W}{\Delta X} = \frac{-W}{X}$$

$$\therefore \quad \ln\left(\frac{W'}{W_0}\right) = \ln\left(\frac{X_0}{X'}\right)$$

$$\therefore \quad W'X' = W_0 X_0$$

$$\therefore \quad W \cdot X = (V_{z0} - u) \cdot L \approx V_{z0} \cdot L \quad \text{since } u << V_{z0}$$

The critical speed comes at $W_c \geq \sqrt{\frac{200kT}{m}}$ for the first time

$$X_{critical} = \frac{V_{z0} L}{\sqrt{\frac{200kT}{m}}}$$

---

let f_1 be the speed it first steps up the potential

$$\therefore \quad \frac{1}{2} m f_1^2 = \frac{1}{2} m W_c^2 - 100 k_B T$$

$$f_1 = \sqrt{W_c^2 - \frac{200 k_B T}{m}}$$

since W_c increases in $2u$

$$\max(W_c) = \sqrt{\frac{200 k_B T}{m}} + 2u \quad \text{(if greater, already passed)}$$

$$\min(W_c) = \sqrt{\frac{200 k_B T}{m}}$$

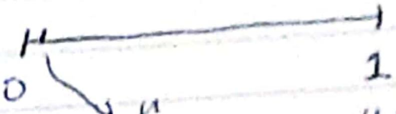
$$\text{let} \quad W_c = \sqrt{\frac{200 k_B T}{m}} + t \cdot 2u \qquad t \in [0, 1)$$

$$\therefore \quad f_1 = \sqrt{4tu\sqrt{\frac{200 k_B T}{m}} + 4t^2u^2}$$

$$f_1 = 2\sqrt{t\frac{\sqrt{\frac{200 k_B T}{m}}}{u} + t^2} \cdot u$$

only if $t \sim \frac{u}{\sqrt{\frac{200k_BT}{m}}}$, $f_1 \sim u$, electron cannot reach barrier again, remains at low KE $\sim mu^2$

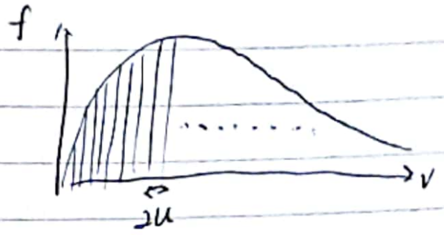
but $t \in [0,1)$



$t >> \frac{u}{\sqrt{\frac{200k_BT}{m}}}$ in most case

$\frac{u}{\sqrt{\frac{200k_BT}{m}}}$ since $u << \sqrt{\frac{k_BT}{m}}$

probability density over $t \in [0,1)$ is almost constant since u very small, it is a "strip" from maxwell distribution



$f(v) \sim$ constant in such interval.

so of $O\left(\frac{u}{\sqrt{\frac{200k_BT}{m}}}\right)$ fraction would not step down barrier repeatedly, the rest and most of them would still reach two walls in a very short period and steps up and down repeatedly.

Phase 2 : after passing barrier, repeatedly steps up and down

since $f_1 = 2\sqrt{t\frac{\sqrt{\frac{200k_BT}{m}}}{u}+t^2}\,u$ for most t

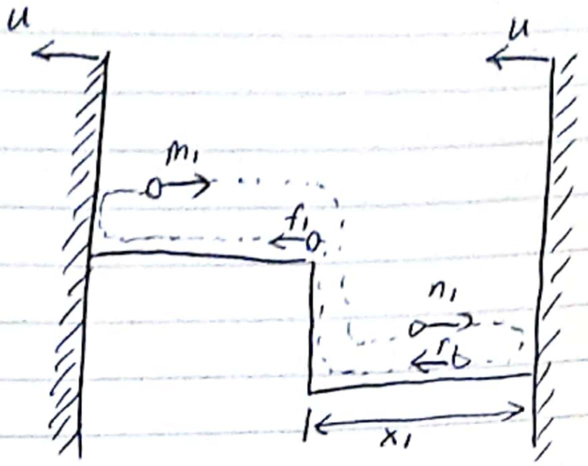
$d_1 \sim \sqrt{\frac{\sqrt{\frac{k_BT}{m}}}{u}} \cdot u >> u$ so will go through many repeated ups and downs before ~~barrier~~ reaching $z=L$.

still in amoving frame with barrier

Let the "cycle" of velocity be

$(f_1 \rightarrow m_1 \rightarrow n_1 \rightarrow r_1) \rightarrow f_2 \rightarrow m_2 \rightarrow \cdots$

as labelled in the diagram below



x_1 is the distance between potential barrier & wall when $v = f_1$ (just pass through barrier).

$$f_1 = f_1$$

$$m_1 = f_1 - 2u$$

$$n_1 = \sqrt{m_1^2 + \frac{200k_BT}{m}}$$

$$r_1 = n_1 + 2u$$

$$f_2 = \sqrt{r_1^2 - \frac{200k_BT}{m}}$$

$$\Rightarrow f_2 = \sqrt{f_1^2 - 4uf_1 + 4u^2 + 4u^2 + \cancel{\frac{200k_BT}{m}} + 4u\sqrt{f_1^2 - 4uf_1 + 4u^2 + \frac{200k_BT}{m}} - \cancel{\frac{20}{m}}}$$

$$= f_1\sqrt{1 + \frac{4u}{f_1}\left(\sqrt{1 - \frac{4u}{f_1} + \frac{4u^2}{f_1^2} + \frac{200k_BT}{mf_1^2}} - 1\right) + \frac{8u^2}{f_1^2}}$$

$u \rightarrow 0$ $\approx f_1\left(1 + \frac{2u}{f_1}\sqrt{\frac{200k_BT}{mf_1^2}}\right)$ to first order

$\therefore \Delta f \approx \dfrac{2u\sqrt{\frac{200k_BT}{m}}}{f}$ after one cycle.

time elapse from $v = f_1$ to $v = f_2$

$$t_{12} = \frac{L - x_1}{f_1 - u}$$

$$+ \frac{\left(\frac{L-x_1}{f_1-u}\right) \cdot u + (L - x_1)}{f_1 - 2u}$$

$$+ \frac{x_1 - u \cdot \left(\frac{L-x_1}{f_1-u} + \frac{L - x_1 + u \cdot \left(\frac{L-x_1}{f_1-u}\right)}{f_1 - 2u} \right)}{n_1 + u}$$

$$+ \frac{1}{r_1} \cdot \left[x_1 - u \cdot \left(\frac{L-x_1}{f_1-u} + \frac{L - x_1 + u\left(\frac{L-x_1}{f_1-u}\right)}{f_1-2u} \right) \right] \cdot \left(1 - \frac{u}{n_1+u}\right)$$

$\because$ $r_1 \sim n_1 >> f_1 >> u$ since r_1, n_1 is down the domain $> \sqrt{\frac{200 k_B T}{m}}$

$$t_{12} \approx \frac{L-x_1}{f_1-u} + \frac{\left(\frac{L-x_1}{f_1-u}\right) u + (L - x_1)}{f_1 - 2u} + 0 + 0$$

$$= \frac{2(L-x_1)}{f_1 - u} \approx \frac{2(L-x_1)}{f_1}$$

$\therefore$ $\Delta t = \frac{2(L-x)}{f}$ in limiting case

$$\therefore \quad \Delta x = -u \Delta t = -\frac{2u(L-x)}{f}$$

$$\frac{df}{dx} = \lim \frac{\Delta f}{\Delta x} = -\frac{\sqrt{\frac{200 k_B T}{m}}}{L - x}$$

$$f_{final} - f_1 = \left(\ln \left(\frac{L - x_{final}}{L - x_1} \right) \right) \cdot \sqrt{\frac{200 k_B T}{m}}$$

$$f_{final} = f_1 + \left(\ln\left(\frac{L - x_{final}}{L - x_1}\right)\right) \cdot \sqrt{\frac{200 k_B T}{m}}$$

$x_{final} = 0$ when wave just passed to $z = L$

$$x_1 = x_{critical} = \frac{V_{z0} L}{\sqrt{\frac{200 k_B T}{m}}}$$

$$f_{final} = f_1 + \sqrt{\frac{200 k_B T}{m}} \ln\left(1 + \frac{x_1}{L - x_1}\right)$$

Note that $\frac{x_1}{L} \sim \frac{1}{\sqrt{200}} = 0.071 < 1$

$$\ln(1+x) \sim x$$

$$f_{final} \approx f_1 + \sqrt{\frac{200 k_B T}{m}} \frac{x_{critical}}{L}$$

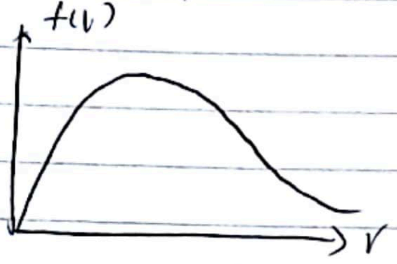
$$f \sim \left(\sqrt{\frac{u}{\sqrt{\frac{k_B T}{m}}}} \cdot \frac{k_B T}{m}\right) \ll V_0$$

$$\approx \frac{V_{critical} \cdot x_{critical}}{L} + 0$$

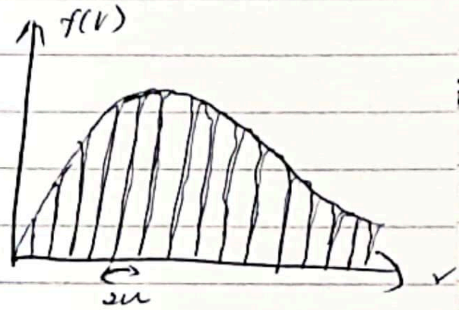
$$= V_0$$

∴ speed unchanged.

initial speed distribution



final



Only a very small portion of electrons slowed down.

$$\overline{V_z^2}_{final} = \overline{V_z^2}_0 \left(1 - O\left(\frac{u}{\sqrt{\frac{k_B T}{m}}}\right)\right)$$

$$P_{final} = P_0 \left(1 - O\left(\frac{u}{\sqrt{\frac{k_B T}{m}}}\right)\right)$$

$P_{final} \to P_0$ if u small enough.

Rethinking adiabatic invariance in part B.

for most $f_1 = \sqrt{4tu\sqrt{\frac{2U_0}{m}} + 4t^2u^2}$

$$f_1 \approx \sqrt{4tu\sqrt{\frac{2U_0}{m}}} >> \sqrt{\frac{KT}{m}} >> u \quad (\text{for large } t)$$

and for large enough V_0 so that $L - x_c$ still large

There is a discontinuous "inflation" of $\int p\,dq$ at first step-up followed by adiabatic invariance

$$\Delta A = (L - x_c) \cdot (f_1 - u)$$

$$= L \cdot \left(1 - \frac{V_0}{\sqrt{\frac{2U_0}{m}}}\right)\left(\sqrt{4tu\sqrt{\frac{2U_0}{m}}} - u\right)$$

$$\frac{V_f}{V_0} = \frac{\Delta A + A}{A} = 1 + \left(\frac{\sqrt{4tu\sqrt{\frac{2U_0}{m}}}}{V_0} - \frac{u}{V_0}\right)\left(1 - \frac{V_0}{\sqrt{\frac{2U_0}{m}}}\right)$$

$$V_f \approx V_0 + \sqrt{4tu\sqrt{\frac{2U_0}{m}}}$$

(for large enough V_0
and large enough t)

Above is for intermediate v_0, if v_0 >sqrt(2U/m) initially, then v_f= v_0

Part B.

from part A, f_1 is the speed after first time step up in the co-moving frame.

$$f_1 = \sqrt{4\sqrt{2}\, t \left(\frac{u}{\sqrt{\frac{k_BT}{m}}\sqrt{\frac{k_BT}{U_0}}}\right)\frac{k_BT}{m} + 4t^2u^2}$$

$$\begin{cases} t \geqslant t^2 \quad \text{for} \quad t \in [0,1) \\ \frac{u}{\sqrt{\frac{k_BT}{m}}\sqrt{\frac{k_BT}{U_0}}}\frac{k_BT}{m} >> \frac{k_BT}{m} >> u^2 \end{cases}$$

$$\therefore f_1 \simeq 2\sqrt{t}\sqrt{u}\sqrt[4]{\frac{2U_0}{m}}$$

If $V_0 \lesssim \sqrt{\frac{2U_0}{m}}$

for the electron to just be able to step down again

$$\frac{x_c}{u} = \frac{2L - x_c}{f_{1c} - u} \qquad (f_{1c} - u) \text{ since in "lab frame"}$$

$$f_{1_c} = \frac{2L - x_c}{x_c} u + u$$

(using $x \cdot V = \text{const.}$)

$$f_{1_c} = \frac{2L - \frac{V_0 L}{\sqrt{\frac{2U_0}{m}}}}{\frac{V_0 L}{\sqrt{\frac{2U_0}{m}}}} u + u = \frac{2\sqrt{\frac{2U_0}{m}}}{V_0} u = 2\sqrt{t_c}\sqrt{u}\sqrt[4]{\frac{2U_0}{m}}$$

$$\sqrt{t_c} = \sqrt{\frac{u}{\sqrt{\frac{2U_0}{m}}}}\,\frac{\sqrt{\frac{2U_0}{m}}}{V_0}$$

$$t_c = \frac{u\sqrt{\frac{2U_0}{m}}}{V_0^2}$$

for $t < t_c$: unable to step down, speed remains at f_1

for $t >> t_c$: discontinuous change in $/pdq$ followed by adiabatic invariance, $V_f = V_0 + 2\sqrt{t}\sqrt{u}\sqrt[4]{\frac{2U_0}{m}}$

let's use t_c as a separating criterion, $t < t_c$

$$V_{final} = f_1 - u = 2\sqrt{t}\sqrt{u}\sqrt[4]{\frac{2U_0}{m}}$$

$$t > t_c : \quad V_{final} = V_0 + 2\sqrt{t}\sqrt{u}\sqrt[4]{\frac{2U_0}{m}}$$

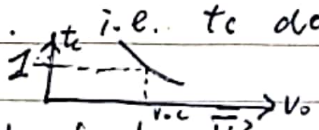
Note that $\frac{\sqrt{200}\, k_BT/m}{V_0^2} >> t_c >> \frac{\sqrt{2}\frac{k_BT}{m}}{V_0^2}$

but $0 \leqslant t \leqslant 1$
compare these boundaries

Hence if V_0 is too low, it will pass barrier when the wave is very close to the other end of wall (x_c very small) thus it is more difficult to chase and then step down again.

Hence, for small enough V_0, all electrons in $[V_0 \pm u]$ not stepping down again

And for large enough V_0, most electrons in $[V_0 \pm u]$ steps down. i.e. t_c decreases to 1 at V_{0c}.



$\Rightarrow V_{0c} = \sqrt{u}\sqrt[4]{\frac{2U_0}{m}}$

Finally, should find $\overline{V_z^2}$

$$\therefore \text{for} \begin{cases} |V_0| \in [0, \sqrt{u}\sqrt[4]{\frac{2U_0}{m}}] & \text{all electrons } \underline{\underline{\text{not}}} \text{ step}^{\text{down}} \\ |V_0| \in [\sqrt{u}\sqrt[4]{\frac{2U_0}{m}}, \sqrt{\frac{2U_0}{m}}] & \text{some step down dependent } t. \\ |V_0| \in [\sqrt{\frac{2U_0}{m}}, \infty] & \text{all step down} \end{cases}$$

Let $f(v_0) = 2\left(\frac{m}{2\pi k_B T}\right)^{\frac{1}{2}} e^{-\frac{mv^2}{2k_BT}}$ $\left(\int_0^\infty f(v_0)\,dv = 1\right)$

find $\overline{V_z}^2$ let dv_0 steps in $2u$ strips

$$\overline{V_z^2}' = \int_0^{\sqrt{u}\sqrt[4]{\frac{2U_0}{m}}} \left(\int_0^1 (f_1-u)^2\,dt\right) f(v_0)\,dv_0$$

$$+ \int_{\sqrt{u}\sqrt[4]{\frac{2U_0}{m}}}^{\sqrt{\frac{2U_0}{m}}} \left(\int_0^{t_c} (f_1-u)^2\,dt + \int_{t_c}^1 \left(v_0 + 2\sqrt{t}\sqrt{u}\sqrt[4]{\frac{2U_0}{m}}\right)^2 dt\right) f(v_0)\,dv_0$$

$$+ \int_{\sqrt{\frac{2U_0}{m}}}^\infty v_0^2 f(v_0)\,dv_0$$

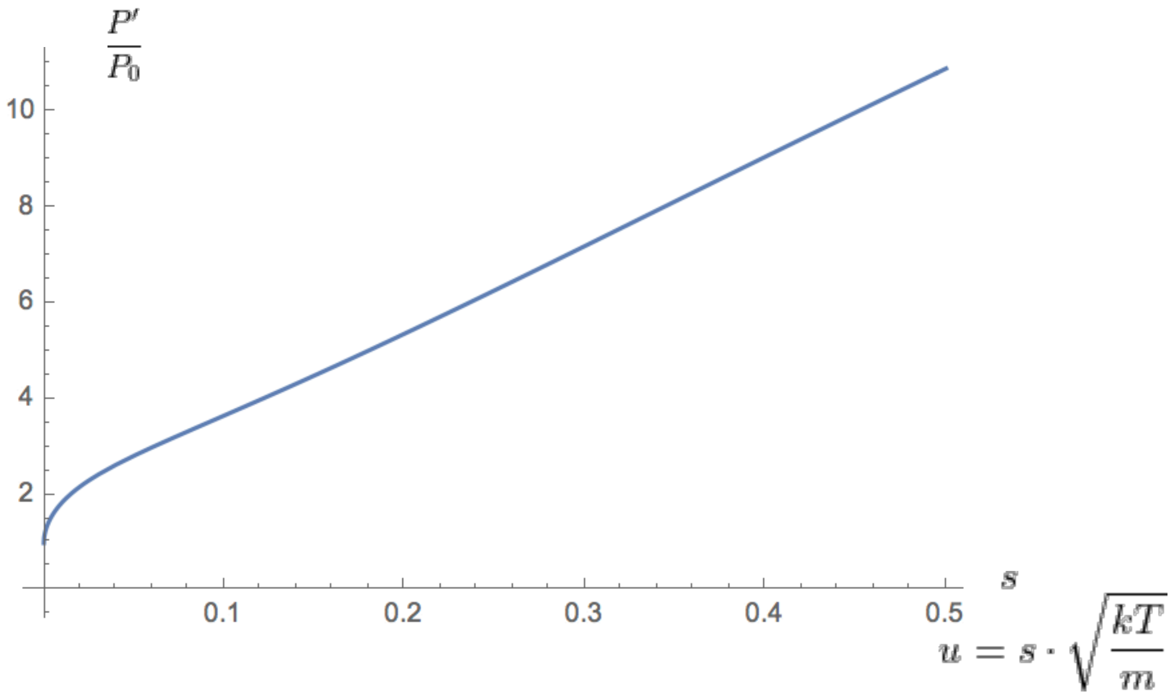
Use Mathematica to integrate

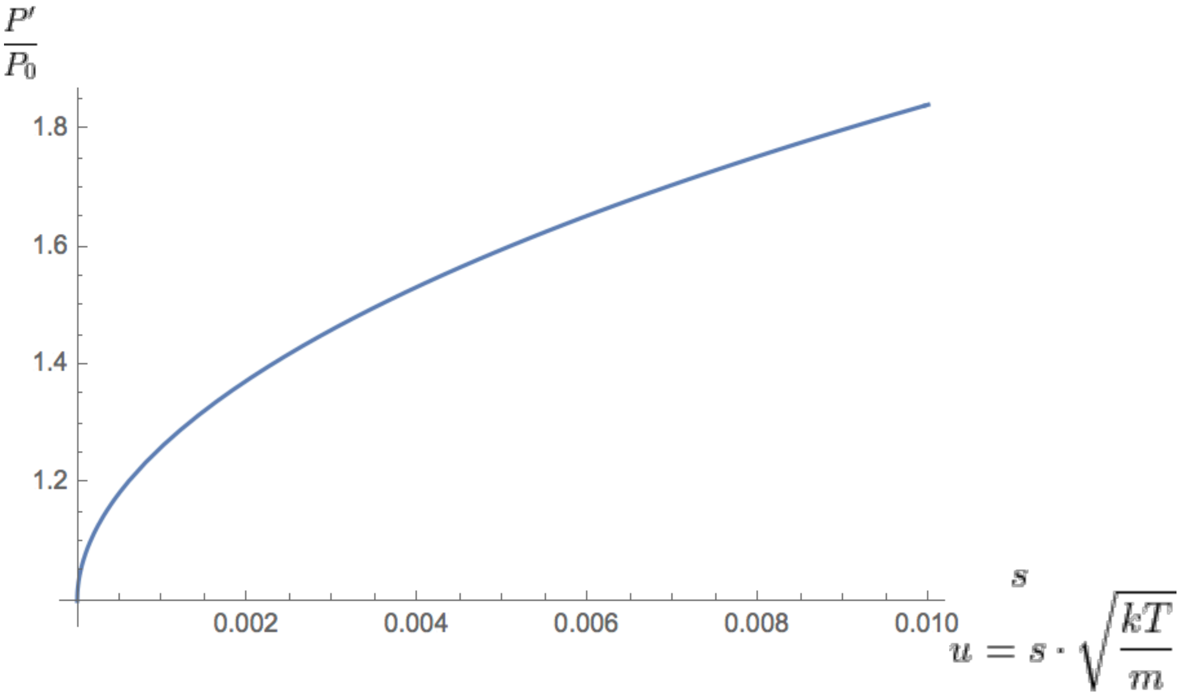
The integral of ratio of mean square speed evaluates to for u being dimensionless with unit of Sqrt[kT/m]

$$
\begin{aligned}
\frac{1}{3}\Bigg(&3 + 22 \times 2^{1/4}\, e^{-5\sqrt{2}\,u} \sqrt{\frac{5}{\pi}}\,\sqrt{u} - \frac{16 \times 2^{1/4}\sqrt{5}\,\sqrt{u}}{e^{100}\sqrt{\pi}} + 60\sqrt{2}\,u - 160 \times 2^{3/4}\, e^{-5\sqrt{2}\,u}\sqrt{\frac{5}{\pi}}\,u^{3/2} + \frac{160\,u^2}{e^{100}\sqrt{\pi}} - \\
&\frac{80\, e^{-5\sqrt{2}\,u}\,u^2}{\sqrt{\pi}} + 6 \times 2^{1/4}\, e^{-5\sqrt{2}\,u}\sqrt{\frac{5}{\pi}}\,u^{5/2} + \frac{\sqrt{\frac{2}{\pi}}\,u^3}{e^{100}} - 30\sqrt{2}\,u\,\text{Erf}[10] + 1600\,u^2\,\text{Erf}[10] - \\
&30\sqrt{2}\,u^3\,\text{Erf}[10] + \left(-3 + 30\sqrt{2}\,u - 8 \times 2^{3/4}\sqrt{5}\,u^{3/2} - 1597\,u^2 + 30\sqrt{2}\,u^3\right)\text{Erf}\left[2^{1/4}\sqrt{5}\,\sqrt{u}\right] - \\
&60\sqrt{2}\,u\,\text{Erfc}[10] + 400\sqrt{\frac{2}{\pi}}\,u^3\,\text{ExpIntegralEi}[-100] - 400\sqrt{\frac{2}{\pi}}\,u^3\,\text{ExpIntegralEi}\left[-5\sqrt{2}\,u\right]\Bigg)
\end{aligned}
$$

Since $$\frac{P'}{P_0} = \frac{\overline{v'^2}}{\overline{v_0^2}}$$

Plot of $\frac{P'}{P_0}$ against s $\left(u = s \cdot \sqrt{\frac{kT}{m}}\right)$





It can be seen that P' approaches P_0 if u is low enough, due to adiabatic invariance;

P' increases for larger u because u is so large that most electrons would be left at a higher KE than initial KE_0 after step-up.
The speed of most electrons after first step-up, f_1, is approximately proportional to $\sqrt{u\sqrt{2U_0/m}}$, hence if u small but not very small, $u \gg \sqrt{kT/m}\sqrt{kT/U_0}$, then $f_1 \gg \sqrt{kT/m}$, those low energy electrons would be left at higher KE than their starting KE_0 and their speed would increase in step-up-step-down after (though not many cycles since u is not very small and the wave is nearly at end for small KE_0)
So for $u \gg \sqrt{kT/m}\sqrt{kT/U_0}$, the final speed of electrons would be proportional to $\sqrt{u\sqrt{2U_0/m}}$ and the pressure proportional to v'^2 would be a linear line proportional to u.
On the whole, it is nearly a linear line with P_0 intercept