Physics Cup Problem 4

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Since there is no interaction among the electrons, we may consider the motion of each electron individually. In particular, we only need to consider motion in the z-direction. Since the magnetic field only affects motion in the xy-plane, we may ignore it.

Throughout this problem, we treat the situation is one-dimensional. (Hence "velocity" always refers to the z-component of the velocity.)

Consider an electron with speed v_0. Typically, $v_0 \sim \sqrt{k_\mathrm{B}T/m}$, meaning that $v_0 \gg u$ and $v_0 < v_w = \sqrt{2U_0/m} = \sqrt{200k_\mathrm{B}T/m}$. We hope to find its final speed v_1 after the shock wave. This will allow us to derive the final velocity distribution from the initial distribution, from which the final pressure p_1 can be calculated.

We divide the motion of the electron into two stages:

1. Before it gains the required kinetic energy U_0 to climb the potential barrier, it bounces elastically between that barrier $z = ut$ and the wall $z = L$. Throughout this process, work done by the potential barrier imparts energy to the electron.

2. Eventually, the electron gains sufficient speed v_w to climb up the potential barrier. From that point on, the electron bounces between the walls $z = 0$ and $z = L$. Every time the electron descends the potential barrier and ascends it, it gains a small amount of energy since the barrier moves. After the shock wave has traveled through the entire space between the walls, the speed of the electron will have increased to v_1.

1 Bouncing between the potential barrier and the wall $z = L$

Consider an electron with speed v bouncing between the potential barrier $z = ut$ and the wall $z = L$. It hits the potential barrier with period $T = 2(L - x)/v$ where $x = ut$. During each collision, the barrier imparts a momentum of $\Delta p = 2mv$ onto the electron. Therefore, the average force that the barrier exerts onto the electron is

$$F = \frac{\Delta p}{T} = \frac{mv^2}{L - x}.$$

The work done by the barrier increases the kinetic energy of the electron:

$$F\,\mathrm{d}x = \mathrm{d}E_k$$

$$\frac{mv^2}{L-x}\,\mathrm{d}x = mv\,\mathrm{d}v$$
$$\frac{\mathrm{d}x}{L-x} = \frac{\mathrm{d}v}{v}$$
$$-\ln(L-x) = \ln(v) + C$$
$$v = \frac{K}{L-x}.$$

The initial condition $v = v_0$ at $x = 0$ gives $K = Lv_0$, so

$$v = \frac{L}{L-x}v_0.$$

Eventually, when v reaches v_w, the electron's kinetic energy reaches U_0, so it climbs the potential barrier and enters the second stage with close to zero speed.[1]

By this point, the potential barrier has moved to $z = x_0$ where

$$v_w = \frac{L}{L-x_0}v_0$$
$$x_0 = \left(1 - \frac{v_0}{v_w}\right)L. \tag{1}$$

2 Bouncing between the walls $z = 0$ and $z = L$

Consider an electron with speed v ($v \gg u$)[2] moving in the positive z-direction starting at $z = 0$. It reaches the potential barrier at position $z = x$ ($x = ut$) in time $t_1 = x/v$, gets accelerated by the potential drop to a speed $v' = \sqrt{v^2 + v_w^2}$, and travels for a time $t_2 = (L-x)/v'$ before bouncing off the wall at $z = L$. The electron then takes time t_2 to meet the potential barrier again, climbs the potential barrier with its speed reduced back to v, and takes time t_1 to arrive back at the wall $z = 0$ before starting a new cycle. This entire process has a period of

$$T = 2(t_1 + t_2) = 2\left(\frac{x}{v} + \frac{L-x}{v'}\right).$$

Each time the electron passes the boundary of the potential barrier, it gains a momentum in the positive z-direction equal to $m(v' - v)$. Therefore, in one period, the potential barrier has imparted an impulse of

$$\Delta p = 2m(v' - v).$$

Thus, the average force that the potential barrier exerts on the electron is

$$F = \frac{\Delta p}{T} = \frac{m(v'-v)}{\frac{x}{v} + \frac{L-x}{v'}}.$$

[1] In particular, the speed of the particle increases by $2u$ every time it collides with the potential barrier, so during stage 1 we always have $v = v_0 + 2ku$ for some non-negative integer k. So, the first time v reaches/exceeds v_w, we typically have $v \approx v_w + u$. After climbing the potential barrier, the speed becomes $v' = \sqrt{(v_w+u)^2 - v_w^2} \approx \sqrt{2v_w u} \gg u$.

[2] Justified by footnote 1.

The work that the potential barrier does on the electron translates to an increase in its kinetic energy:

$$\begin{aligned}
F\,\mathrm{d}x &= \mathrm{d}E_k \\
\frac{m(v'-v)}{\frac{x}{v}+\frac{L-x}{v'}}\,\mathrm{d}x &= mv\,\mathrm{d}v \\
\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{v'-v}{x+\frac{v}{v'}\left(L-x\right)}.
\end{aligned} \tag{2}$$

This differential equation is difficult to solve exactly, so we'll make approximations. Assume $v \leq v_0$, which is justified by our final solution (3). Therefore, approximating $v' = \sqrt{v^2+v_w^2} \approx v_w$ causes a relative error of $\sim (v/v_w)^2 \leq (v_0/v_w)^2 \sim 1/200 \ll 1$. Thus, the numerator of the RHS of (2) can be well-approximated as $v_w - v$. We approximate the denominator $x + \frac{v}{v'}\left(L-x\right) \approx x$, which causes a relative error on the order of

$$\frac{v}{v'}\frac{L-x}{x} \leq \frac{v_0}{v_w}\frac{L-x_0}{x_0} \approx \frac{v_0}{v_w}\frac{L-x_0}{L} = \left(\frac{v_0}{v_w}\right)^2 \ll 1,$$

where we've used (1). Hence, (2) becomes

$$\frac{\mathrm{d}v}{\mathrm{d}x} \approx \frac{v_w - v}{x}$$

with high accuracy.[3] We now solve the differential equation:

$$\begin{aligned}
-\frac{\mathrm{d}v}{v_w - v} &= -\frac{\mathrm{d}x}{x} \\
\ln(v_w - v) &= -\ln(x) + C \\
v_w - v &= \frac{K}{x} \\
v &= v_w - \frac{K}{x}.
\end{aligned}$$

We have the initial condition $v = 0$ at $x = x_0 = \left(1 - \frac{v_0}{v_w}\right)L$, which gives $K = (v_w - v_0)L$. Therefore,

$$v = v_w - \frac{L}{x}(v_w - v_0). \tag{3}$$

When the shock wave reaches the wall at $z = L$, we have $x = L$. Substituting this into (3) gives the final speed of the electron

$$v_1 = v_0.$$

[3] In relative terms, the numerator is underestimated by about $v^2/2v_w^2$, whose average is $v_0^2/6v_w^2$ since v changes approximately linearly from 0 to v_0 as x changes from x_0 to L (see (3)). The denominator is underestimated by about $v(v_0 - v)/v_w^2$, whose average is also $v_0^2/6v_w^2$. $\mathrm{d}v/\mathrm{d}x \approx v_w/L$ is approximately constant, so these two underestimations on average cancel each other out as we integrate with respect to x. Therefore, in the end, the relative error in v_1 and also p_1 is at most $\sim (v_0/v_w)^3 \sim 10^{-3}$.

3 The final pressure

Since the speed of every electron remains unchanged after the shock wave, the velocity distribution of the electrons remains unchanged. Therefore, the final pressure of the electron gas is equal to its initial pressure:

$$p_1 = p_0.$$