Physics Cup 2021, Problem 4

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Let us first address the inequality $B \gg mk_BT/\hbar e$. It can be written as $\hbar\omega_c \gg k_BT$, where $\omega_c = eB/m$ — the cyclotron frequency. From here it is evident that with regard to the transversal motion of the electrons (that is, in a plane parallel to the walls) they are confined in a small region of size $l \approx \sqrt{\hbar/eB}$ well within the quantum limit. That is because the energy levels of an electron in a magnetic field are (the transverse part)

$$E_n = \hbar\omega_c(n + 1/2), \tag{1}$$

and the probabilities of an electron being at level $n = 1$ and higher are proportional to the factor

$$\exp(-\hbar\omega_c/k_BT) \ll 1. \tag{2}$$

That, combined with the sparsity of the gas (here we assume the gas is non-degenerate, so we can use the Maxwell's distribution), means that as soon as the field is applied, the electrons decouple from each other. There are no collisions between electrons, so the root mean square velocities in all cardinal directions don't equalize, as they would without the strong field (the electrons don't thermalize, as one could say). As for the z-axis motion, the field doesn't affect it at all, so the only factor affecting this motion is the shock wave potential.

So we begin by examining the process of interaction of a single electron with the shock wave. Take an electron with a velocity magnitude v_0 along the z-axis. Since $U_0 \gg k_BT$, it is highly likely that v_0 would not be enough to jump on the barrier the first time the electron hits it, so the electron bounces back, gaining additional velocity $2u$. Since $u \ll \sqrt{2k_BT/m}$, it would probably be still not enough, and on the second bounce another $2u$ is added. So on the n-th bounce the velocity would be

$$v_n = v_0 + 2un. \tag{3}$$

This repeats until the velocity reaches the value greater than

$$\sqrt{\frac{2U_0}{m}} = 10\sqrt{\frac{2k_BT}{m}} \equiv 10v_T, \tag{4}$$

where hereafter we use the notation $v_T \equiv \sqrt{2k_BT/m}$. We have an inequality

$$v_0 + 2un > 10v_T, \tag{5}$$

the integer solution to which is

$$n = \left\lfloor \frac{10v_T - v_0}{2u} \right\rfloor + 1. \tag{6}$$

We assume that for the most electrons initially present this is a very large number. With this velocity the electron jumps on the barrier, losing U_0 of its energy. The velocity of the electron will be

$$v_f = \sqrt{v_n^2 - 100v_T^2} = \sqrt{(v_n - 10v_T)(v_n + 10v_T)} \approx \sqrt{20v_T(v_n - 10v_T)}. \tag{7}$$

The difference between the velocities is equal to

$$v_n - 10v_T = v_0 + 2u\left(\left\lfloor \frac{10v_T - v_0}{2u} \right\rfloor + 1\right) - 10v_T = 2u\left(\left\lfloor \frac{10v_T - v_0}{2u} \right\rfloor - \frac{10v_T - v_0}{2u} + 1\right) = 2u\left(1 - \left\{\frac{10v_T - v_0}{2u}\right\}\right). \tag{8}$$

The function $v_f^2(v_0)$ is, as follows from (7) and (8), is a sawtooth function with period $2u$ which spans the interval of v_0 from 0 to $10v_T$ (at higher velocities the electron will traverse the barrier without bouncing off of it once). Since $u \ll v_T$, we can average the sawtooth function to a value which is half its maximum:

$$v_f^2 \approx \frac{1}{2} \cdot 20v_T \cdot 2u = 20v_Tu. \tag{9}$$

That, in turn, means that $v_f \gg u$, so even after jumping on the barrier and losing most of its energy, the electron is still moving much faster than the barrier. This fact allows us to use the adiabatic theorem: if a system is subjected to slow varying conditions and experiences periodic motion, then the phase space volume of the system will remain constant. In this case, the initial phase space trajectory is a rectangle with length L and width $2p_0 = 2mv_0$. During the first part of the process, the rectangle shrinks coordinate-wise and expands momentum-wise so that the area stays constant. When the half-width of the rectangle reaches the value of $10mv_T$, a second rectangle emerges in the region which wasn't accessible to the electron before (behind the wave front). In the end, this second rectangle will be of length L, and the first one will disappear. During all of this, the total area of the two rectangles will be the same as the area of the initial rectangle. But that means that the initial and final rectangles, having equal lengths of L, should have equal widths, so the actual final velocity of the electron will be the same as its initial one. Since the pressure of a gas depends on the mean-square velocity of its particles and on their number density, and none of those change, then the pressure will be the same as before the shock wave:

$$\boxed{p = p_0.} \tag{10}$$

However, we must also include the case when the adiabatic approximation doesn't work. We know that the approximation holds when

$$\left|\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right| \ll \frac{\lambda}{T}, \tag{11}$$

where λ — a parameter which defines the external conditions (in this case, one could use the position of the shock wave), and T is the period of the system under given conditions. Let us determine the velocities which satisfy this inequality. By λ we take the distance x between the shock wave front and the second wall. Using the adiabatic invariant mentioned above, we can determine x at the moment of the jump:

$$x = L/k, \tag{12}$$

where $k = \sqrt{U_0/k_BT} = 10$. The period at the time of the jump is equal to

$$T = \frac{x}{kv_T} + \frac{L-x}{v_f} \sim \frac{L}{k^2 v_T} + \frac{L}{\sqrt{kv_T u}} \approx \frac{L}{\sqrt{kv_T u}}, \tag{13}$$

since the denominator of the second fraction is much smaller than of the first. The rate of change of x is, obviously, the wave's velocity, u. Hence, we come to an inequality

$$u \ll \frac{x\sqrt{kv_T u}}{L} = \sqrt{\frac{v_T u}{k}}, \tag{14}$$

which can be written as

$$\boxed{u \ll \frac{v_T}{k} \sim \sqrt{\frac{k_BT}{U_0}}\sqrt{\frac{k_BT}{m}}.} \tag{15}$$

If the opposite holds, then we can assume the electron maintains the velocity v_f from (9) until the end. To determine the pressure we calculate the force exerted by electrons that have an initial velocity magnitude near v_0. The force is momentum change with each collision divided by the time between the collisions times the number of electrons with this velocity, which is equal to the Maxwell's distribution of the initial velocities v_0 (because, as mentioned earlier, the electrons don't thermalize, so the initial distribution stays until the very end):

$$dF = \frac{2mv_f(v_0)}{2L/v_f(v_0)} \cdot n(v_0)\,dv_0 = \frac{mv_f^2(v_0)}{L} \cdot 2N\sqrt{\frac{m}{2\pi k_BT}} \exp\left(-mv_0^2/2k_BT\right) dv_0 = \frac{2Nmv_f^2(v_0)}{Lv_T\sqrt{\pi}} \exp\left(-v_0^2/v_T^2\right) dv_0. \tag{16}$$

Here N is the total number of electrons, and the factor of 2 is added to get the 1-sided distribution of the velocity magnitude. Since $v_T \ll 10v_T$, we can expand the integration interval to $(0, +\infty)$, which leaves us with

$$F = \frac{40mNv_T u}{Lv_T\sqrt{\pi}} \int_0^{+\infty} \exp\left(-v_0^2/v_T^2\right) dv_0 = \frac{40mNu}{L\sqrt{\pi}} \cdot v_T \frac{\sqrt{\pi}}{2} = \frac{20mNv_T u}{L}. \tag{17}$$

Technically, we also have to include the contribution of initial velocities higher than $10v_T$ in our force. But since they are much higher than the thermal velocity, the distribution at such velocities becomes vanishingly small, so we neglect it. If the walls' area is S, then the pressure would be

$$p = \frac{F}{S} = 20mv_T u \cdot \frac{N}{LS}. \tag{18}$$

The last fraction in the equation above is the electrons' number density n, which is directly linked to the initial pressure by a well known formula $p_0 = nk_BT$. Excluding n, we get the final answer:

$$\boxed{p = 20mv_T u \cdot \frac{p_0}{k_BT} = 20p_0 u\sqrt{\frac{2m}{k_BT}}.} \tag{19}$$