Physics Cup 2021, Problem 5

Nikita Semenin, MIPT

Consider the steady state of the fields, when at each point the amplitude of the wave doesn't change in time for both harmonics. We will denote the amplitudes arriving at the point C with a tilde (see fig. 1). From the statement

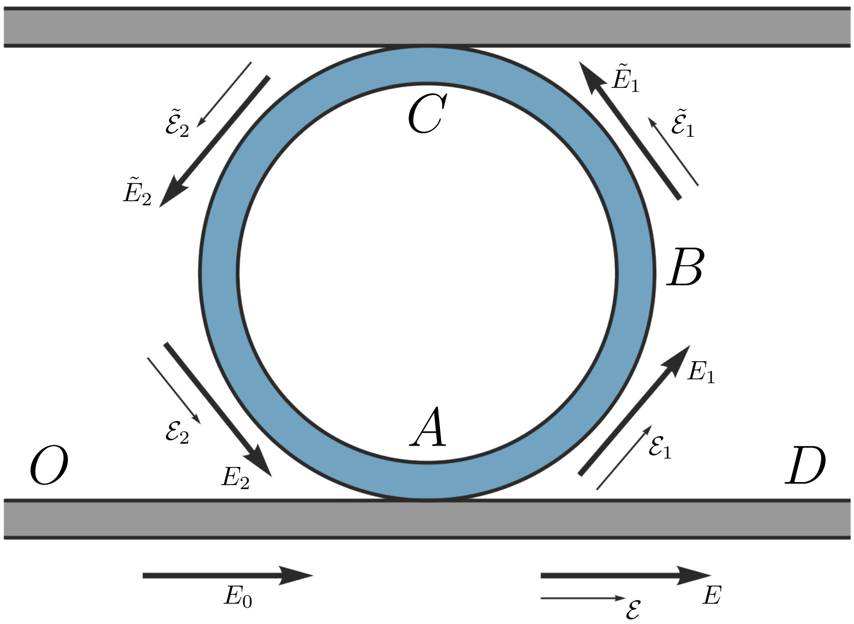

Figure 1

of the problem follows that

$$\tilde{\mathcal{E}}_1 = \mathcal{E}_1 + \delta E_1^2. \tag{1}$$

At the same time, the total optical energy of the two harmonics must be conserved (assuming the optical loss in the fibre itself is negligible). The intensity of the wave is proportional to the value nE^2, where n is the refractive index of the fibre, and E is the amplitude. Since n for both harmonics is the same, the conservation of energy can be written as

$$E_1^2 + \mathcal{E}_1^2 = \tilde{E}_1^2 + \tilde{\mathcal{E}}_1^2, \tag{2}$$

from which follows, using (1):

$$\tilde{E}_1 = E_1\sqrt{1 - 2\delta\mathcal{E}_1 - \delta^2 E_1^2} \approx E_1(1 - \delta\mathcal{E}_1). \tag{3}$$

After travelling through the point C the amplitudes are multiplied by the respective coupling coefficients, since there is no input in the second fibre:

$$\tilde{\mathcal{E}}_2 = \sqrt{1-\alpha^2}\,\tilde{\mathcal{E}}_1 = \sqrt{1-\alpha^2}\left(\mathcal{E}_1 + \delta E_1^2\right); \tag{4}$$

$$\tilde{E}_2 = \sqrt{1-\alpha}\,\tilde{E}_1 = \sqrt{1-\alpha}\,(1 - \delta\mathcal{E}_1)E_1. \tag{5}$$

Here we assume that there is no phase shift induced through coupling. Analogously, the amplitudes arriving at the point A are equal

$$\mathcal{E}_2 = \tilde{\mathcal{E}}_2 + \delta\tilde{E}_2^2 = \sqrt{1-\alpha^2}\left(\mathcal{E}_1 + \delta E_1^2\right) + (1-\alpha)(1 - 2\delta\mathcal{E}_1)\delta E_1^2; \tag{6}$$

$$E_2 = \tilde{E}_2(1 - \delta\tilde{\mathcal{E}}_2) = \sqrt{1-\alpha}\,(1 - \delta\mathcal{E}_1)\Big(1 - \delta\sqrt{1-\alpha^2}\left(\mathcal{E}_1 + \delta E_1^2\right)\Big)E_1. \tag{7}$$

At the point A there are two inputs with respect to the main harmonic: from the ring and straight fibres. For consistency, the amplitude coupled from them into the ring fibre must equal E_1. Since the coherence length of the laser is large, these two input waves are coherent, therefore we must add amplitudes rather than intensities:

$$E_1 = \sqrt{\alpha}E_0 + \sqrt{1-\alpha}E_2. \tag{8}$$

There is no phase factor here, because the length of the fibre is an integer multiple of the wavelength, so this phase is proportional to 2π. Using (7), E_1 equals

$$E_1 = \sqrt{\alpha}E_0 + (1-\alpha)(1-\delta\mathcal{E}_1)\Big(1-\delta\sqrt{1-\alpha^2}\big(\mathcal{E}_1+\delta E_1^2\big)\Big)E_1. \tag{9}$$

The same argument could be made for the second harmonic: the phase would be proportional to 4π, which would also add no phase factor. Therefore, $\mathcal{E}_1$ equals

$$\mathcal{E}_1 = \sqrt{1-\alpha^2}\mathcal{E}_2 = (1-\alpha^2)(\mathcal{E}_1+\delta E_1^2) + \delta(1-\alpha)\sqrt{1-\alpha^2}(1-2\delta\mathcal{E}_1)E_1^2. \tag{10}$$

The last equation can be solved in terms of $\mathcal{E}_1$:

$$\mathcal{E}_1 = \frac{(1-\alpha^2)+(1-\alpha)\sqrt{1-\alpha^2}}{\alpha^2+2\delta^2(1-\alpha)\sqrt{1-\alpha^2}E_1^2}\,\delta E_1^2. \tag{11}$$

The output wave has the amplitude

$$\mathcal{E} = \frac{\sqrt{\alpha^2}}{\sqrt{1-\alpha^2}}\mathcal{E}_1 = \frac{\alpha\big(\sqrt{1-\alpha^2}+(1-\alpha)\big)}{\alpha^2+2\delta^2(1-\alpha)\sqrt{1-\alpha^2}E_1^2}\,\delta E_1^2. \tag{12}$$

We can revert this equation and solve for E_1:

$$E_1^2 = \frac{\alpha^2\mathcal{E}}{\alpha\delta(\sqrt{1-\alpha^2}+1-\alpha)-2\delta^2\mathcal{E}(1-\alpha)\sqrt{1-\alpha^2}}. \tag{13}$$

Since $\alpha \ll 1$, we can approximate the expression:

$$E_1^2 \approx \frac{\alpha^2\mathcal{E}}{2\alpha\delta-2\delta^2\mathcal{E}} = \frac{\alpha}{2\delta}\,\frac{\alpha\mathcal{E}}{\alpha-\delta\mathcal{E}}. \tag{14}$$

From the statement of the problem $\delta\mathcal{E}_1 \ll 1$. Therefore, $\alpha\delta\mathcal{E}_1 \ll \alpha$. Since $\mathcal{E} \approx \alpha\mathcal{E}_1$, then $\delta\mathcal{E} \ll \alpha$, so we can further approximate the denominator:

$$E_1^2 \approx \frac{\alpha\mathcal{E}}{2\delta}. \tag{15}$$

Now we turn to the equation (9). We can again simplify (11) using $\alpha \ll 1$:

$$\mathcal{E}_1 \approx \frac{2\delta E_1^2}{\alpha^2+2\delta^2E_1^2} \approx \frac{2\delta E_1^2}{\alpha^2}. \tag{16}$$

From here it is evident that $\delta E_1^2 \ll \mathcal{E}_1$, so the whole expression (9) can be approximated as

$$E_1 \approx \sqrt{\alpha}E_0 + (1-\alpha)\left(1-\frac{4\delta^2E_1^2}{\alpha^2}\right)E_1 \approx \sqrt{\alpha}E_0 + E_1 - \alpha E_1 - \frac{4\delta^2E_1^3}{\alpha^2}. \tag{17}$$

Thus we have arrived at a cubic equation

$$\frac{4\delta^2}{\alpha^2}\,E_1^3 + \alpha E_1 - \sqrt{\alpha}E_0 = 0. \tag{18}$$

Using (15) we can re-write the equation in terms of $\mathcal{E}$:

$$\frac{\sqrt{2\delta}}{\alpha}\,\mathcal{E}^{3/2} + \frac{\alpha}{\sqrt{2\delta}}\,\sqrt{\mathcal{E}} - E_0 = 0. \tag{19}$$

Using the substitution $x = \sqrt{\mathcal{E}}$, we have the equation

$$\frac{\sqrt{2\delta}}{\alpha}\,x^3 + \frac{\alpha}{\sqrt{2\delta}}\,x - E_0 = 0, \tag{20}$$

the solution to which is a function of α. Finding the maximum of this function is essentially the goal of the problem. However, instead of solving the equation directly, we make use of the fact that at the extremal point of $x(\alpha)$ its derivative is equal to zero. So we can differentiate both sides of the equation with respect to α and put $x'(\alpha_m) = 0$ to get another equation, which is much easier to solve:

$$-\frac{\sqrt{2\delta}}{\alpha_m^2}\,x_m^3 + \frac{1}{\sqrt{2\delta}}\,x_m = 0. \tag{21}$$

The solution is found as

$$x_m = \frac{\alpha_m}{\sqrt{2\delta}}. \tag{22}$$

Now we substitute x_m back into (20) to get

$$\frac{\sqrt{2\delta}}{\alpha_m}\left(\frac{\alpha_m}{\sqrt{2\delta}}\right)^3 + \frac{\alpha_m}{\sqrt{2\delta}}\left(\frac{\alpha_m}{\sqrt{2\delta}}\right) - E_0 = 0, \tag{23}$$

therefore

$$\boxed{\alpha_m = \sqrt{\delta E_0} = \sqrt{\delta\sqrt{I_0}}.} \tag{24}$$

From this we can easily get the optimal second harmonic intensity:

$$\boxed{\mathcal{I}_m = x_m^4 = \frac{I_0}{4}.} \tag{25}$$

We might also check whether the assumption $\delta\mathcal{E}_1 \ll 1$ is justified. For the optimal value of α

$$\delta\mathcal{E}_1 = \frac{\delta\mathcal{E}_m}{\alpha_m} = \frac{\sqrt{\delta E_0}}{2}. \tag{26}$$

Therefore, when $\sqrt{\delta E_0} \ll 1$, the assumption is justified, which would also automatically justify the assumption $\alpha \ll 1$.