Let us begin by proving the fact described in the first hint. Actually, I will prove a somewhat more general statement. However, before the statement and proof of the theorem I want to prove, let us state the conditions under which we are working.

Let $\Omega$ be a lossless optical device that connects two media, $\alpha$ and $\beta$. Let an electromagnetic wave $A$ that comes from $\alpha$ and enters $\Omega$ be split into two light waves, one ($A'$) that goes back towards $\alpha$ and the other ($A''$) towards $\beta$; let the phase difference between $A''$ and $A'$ be $\Delta\phi_A$. Similarly, let an electromagnetic wave $B$ that comes from $\beta$ and enters $\Omega$ be split into two light waves, one ($B'$) that goes back towards $\beta$ and the other ($B''$) towards $\alpha$; let the phase difference between $B''$ and $B'$ be $\Delta\phi_B$.

**Theorem**

The sum of the phase differences obeys

$$\Delta\phi_A + \Delta\phi_B = \pi. \tag{1}$$

*Proof.* This proof will consist of two parts: first, I will study the distribution of energy in the situations described above, then I will use the principle of superposition to study the distribution of energy in another situation.

- Due to energy conservation, the intensities of the different waves in the situations described above obey

$$\begin{aligned} I_A &= I_{A'} + I_{A''}; \\ I_B &= I_{B'} + I_{B''}. \end{aligned} \tag{2}$$

- Let us now superpose the above two situations and set up the electromagnetic waves $A$ and $B$ such that they are both of the same frequency and the wave $A'$ is in phase with the wave $B''$. In this situation, the phase difference between $A''$ and $B'$ is given by

$$\phi_{A''} - \phi_{B'} = (\phi_{A'} + \Delta\phi_A) - (\phi_{B''} - \Delta\phi_B) = \Delta\phi_A + \Delta\phi_B. \tag{3}$$

Since $A'$ and $B''$ are in phase, they interfere constructively, so that their amplitudes add; as a consequence, the energy carried by the new wave formed is

$$\left(\sqrt{I_{A'}} + \sqrt{I_{B''}}\right)^2 = I_{A'} + I_{B''} + 2\sqrt{I_{A'} I_{B''}}.$$

The energy carried by the other outgoing wave is given by (using the vector sum amplitude formula)

$$I_{A''} + I_{B'} + 2\sqrt{I_{A''} I_{B'}} \cos(\Delta\phi_A + \Delta\phi_B).$$

Energy conservation requires that

$$I_A + I_B = I_{A'} + I_{B''} + 2\sqrt{I_{A'} I_{B''}} + I_{A''} + I_{B'} + 2\sqrt{I_{A''} I_{B'}} \cos(\Delta\phi_A + \Delta\phi_B), \tag{4}$$

or, using Eq. (2),

$$\cos(\Delta\phi_A + \Delta\phi_B) = -\sqrt{\frac{I_{A'}}{I_{A''}} \frac{I_{B''}}{I_{B'}}}. \tag{5}$$

Due to symmetry,

$$\frac{I_{A'}}{I_{A''}} = \frac{I_{B'}}{I_{B''}},$$

so that

$$\cos(\Delta\phi_A + \Delta\phi_B) = -1 \Longrightarrow \Delta\phi_A + \Delta\phi_B = \pi, \tag{6}$$

which was to be proven.

□

In our situation, $\Omega$ can be considered either of the junctions $A$ or $C$.

Let $E_D$ be the amplitude of the base-frequency radiation that leaves the junction $A$ along the cable $AD$, and let $E_B$ be the amplitude of the base-frequency radiation that leaves the junction $A$ along the semicircle $ABC$; this intensity is approximately constant inside the circular resonator (because $\alpha \ll 1$ and $\delta\varepsilon \ll 1$). Also, let $E_G$ be the amplitude of the base-frequency radiation that leaves the resonator through the cable $CG$.

*Note:* I will further neglect the coefficient $C$ of proportionality in the relationship $I = CE^2$, *i.e.* I will work in units where it is equal to 1.

The phase of the light waves that go around the resonator and reach the junction $A$ again will be the same as the phase of the light waves that propagate along the fibre from $O$, because the length of the resonator is an integer number of wavelengths of the light (through the definition of a resonator). Because of this, using the theorem proven earlier, there will be a phase shift of $\pi$ when the radiation inside the resonator passes through the junction $A$ into the cable $AD$. Hence, this radiation will interfere destructively with the rest of the radiation $E_0$ propagating along the cable.

Since the energetic transmittance of the junction is $\alpha$, the fraction by which the transmitted electric field *amplitude* will be smaller is $\sqrt{\alpha}$; hence,

$$E_D = E_0 - \sqrt{\alpha} E_B. \tag{7}$$

*Note:* In the above equation I have neglected the change of $E_0$; because a part $\alpha$ of $I_0$ is transmitted towards $B$, the part of the electric field that passes along $AD$ is actually $\sqrt{1-\alpha}E_0$, not $E_0$. However, since $\alpha \ll 1$, the difference is insignificant.

Similarly, the field $E_G$ is caused by the transmission of energy through the junction $C$; more precisely (assuming that $E_B$ is, as stated above, approximately constant), we see that

$$E_G = \sqrt{\alpha} E_B. \tag{8}$$

The total intensity of the base-frequency radiation that leaves the system is, as such,

$$I_{tot} = E_D^2 + E_G^2 \Longrightarrow I_{tot} = E_0^2 - 2\sqrt{\alpha}E_0 E_B + 2\alpha E_B^2. \tag{9}$$

Let us now turn to the double-frequency radiation. Due to symmetry, the same intensity $\mathcal{I}$ will flow out through both of the cables $AD$ and $CG$. Hence, the total outgoing radiative power of the double-frequency radiation is $2\mathcal{I}$, and the energy conservation equation for the whole system reads

$$I_0 = I_{tot} + 2\mathcal{I}. \tag{10}$$

Over one semicircle, the amplitude of the double-frequency radiation increases by $\Delta\varepsilon = \delta E_B^2$. In the steady state, exactly this excess energy will be radiated at the junctions (if it weren't, then there would be a long-term change in the amplitude of the radiation, which contradicts the steady-state condition). Hence,

$$\mathcal{I} = \Delta(\varepsilon^2) = 2\varepsilon\Delta\varepsilon \Longrightarrow \varepsilon = \frac{\mathcal{I}}{2\Delta\varepsilon}. \tag{11}$$

However, we also know the rate at which the light energy is being transmitted through the junctions:

$$\mathcal{I} = \alpha^2\varepsilon^2. \tag{12}$$

Combining Eqs. (11) and (12), we find that

$$\mathcal{I} = \alpha^2\left(\frac{\mathcal{I}}{2\Delta\varepsilon}\right)^2 \Longrightarrow \mathcal{I} = \left(\frac{2\Delta\varepsilon}{\alpha}\right)^2 = \left(\frac{2\delta E_B^2}{\alpha}\right)^2 \Longrightarrow E_B = \sqrt{\frac{\alpha}{2\delta}}\mathcal{I}^{1/4}. \tag{13}$$

Using the expression above for $E_B$, Eq. (10) becomes

$$\begin{aligned} E_0^2 &= E_0^2 - 2\sqrt{\alpha}E_0E_B + 2\alpha E_B^2 + 2\mathcal{I} \\ \iff E_0 &= \frac{\alpha}{\sqrt{2\delta}}\mathcal{I}^{1/4} + \frac{\sqrt{2\delta}}{\alpha}\mathcal{I}^{3/4}. \end{aligned} \tag{14}$$

Using the AM $\geq$ GM inequality, the term on the right obeys

$$\frac{\alpha}{\sqrt{2\delta}}\mathcal{I}^{1/4} + \frac{\sqrt{2\delta}}{\alpha}\mathcal{I}^{3/4} \geq 2\sqrt{\mathcal{I}}, \tag{15}$$

so that

$$E_0 \geq 2\sqrt{\mathcal{I}} \Longrightarrow \mathcal{I}_m = \frac{E_0^2}{4} \Longrightarrow \boxed{\mathcal{I}_m = \tfrac{I_0}{4}}. \tag{16}$$

Indeed, for the equality condition in the inequality, which is

$$\frac{\alpha}{\sqrt{2\delta}}\mathcal{I}^{1/4} = \frac{\sqrt{2\delta}}{\alpha}\mathcal{I}^{3/4}, \tag{17}$$

$\mathcal{I}$ indeed takes the maximum value shown above.

Under these conditions, the equality condition (17) tells us that

$$\alpha^2 = 2\delta\sqrt{\mathcal{I}_m} = \delta E_0 \Longrightarrow \boxed{\alpha_m = \sqrt{\delta E_0}}. \tag{18}$$