Physics Cup Problem 5

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The circumference of the circular fiber is an integer multiple of the wavelength of both the main-frequency wave and the double-frequency wave. Therefore, both waves are standing waves and will not destructively interfere with themselves.

Note that when the laser is first shot into the system, the amplitude E of the main-frequency light in the circular fiber will continuously increase until the rate of amplitude loss[1] (at junctions A and C and due to second-harmonic generation) becomes large enough to match the rate of amplitude gain[2] (due to energy input from the laser through junction A). The amplitude $\mathcal{E}$ of the double-frequency light will also increase until the rate of amplitude loss[3] (at junctions A and C) becomes large enough to cancel out the rate of amplitude gain[4] (due to second-harmonic generation). Therefore, after some time, the system will reach steady state where the amplitude of the wave at every point in the system remains constant.

At steady state, the relative change in amplitude of a wave traveling along the circular fiber is small as $\alpha, \alpha^2, \delta\mathcal{E} \ll 1$. Therefore, let I_c and E_c be the average intensity and amplitude of the main-frequency wave in the circular fiber ($I_c = E_c^2$), and let $\mathcal{I}_c$ and $\mathcal{E}_c$ be the average intensity and amplitude of the double-frequency wave in the circular fiber ($\mathcal{I}_c = \mathcal{E}_c^2$).

As a beam of main-frequency photons travels along the circular fiber, 2 photons are converted to 1 photon of twice the energy traveling at the same speed, so the total light intensity remains constant. Thus, along arc ABC, the total change in the intensities of the main-frequency and double-frequency waves is 0:

$$\begin{aligned} \Delta\left(E_c^2 + \mathcal{E}_c^2\right) &= 0 \\ 2E_c\Delta E_c + 2\mathcal{E}_c\Delta\mathcal{E}_c &= 0 \\ \Delta E_c &= -\mathcal{E}_c\Delta\mathcal{E}_c/E_c = -\delta\mathcal{E}_c E_c, \end{aligned}$$

where we've used $\Delta\mathcal{E}_c = \delta E_c^2$.

As the main-frequency wave travels the circumference of the circular fiber, the amplitude is increased by $\sqrt{\alpha I_0}$ at junction A,[5] decreased by $\alpha E_c/2$ at each junction (A and C),[6] and

[1] This rate increases with increasing E.

[2] This rate is constant.

[3] This rate increases with increasing $\mathcal{E}$.

[4] This rate is constant for constant E.

[5] Due to the long coherence length, the wave leaked into the circular fiber will always be in phase with the existing wave. Thus, amplitude is additive.

[6] $E' = \sqrt{1-\alpha}E \approx (1-\alpha/2)E$, so $\Delta E = -\alpha E/2$.

decreased by $2|\Delta E_c|$ (i.e., increased by $2\Delta E_c$) due to second-harmonic generation. Thus,

$$\begin{aligned} \sqrt{\alpha I_0} - \alpha E_c + 2\Delta E_c &= 0 \\ \sqrt{\alpha I_0} &= \alpha E_c + 2\delta\mathcal{E}_c E_c. \end{aligned} \tag{1}$$

As the double-frequency wave travels the circumference of the circular fiber, the amplitude is increased by $2\Delta\mathcal{E}_c = 2\delta E_c^2$ from second-harmonic generation and decreased by $\alpha^2\mathcal{E}_c/2$ through each junction (A and C).[7] Thus,

$$2\delta E_c^2 = \alpha^2\mathcal{E}_c. \tag{2}$$

Now, (1) gives

$$\begin{aligned} \sqrt{\alpha I_0} &\geq 2\sqrt{2\alpha\delta\mathcal{E}_c E_c^2} && \text{(AM-GM)} \\ &= 2\sqrt{\alpha^3\mathcal{E}_c^2} && \text{(by (2))} \\ I_0 &\geq 4\alpha^2\mathcal{E}_c^2 = 4\mathcal{I}, \end{aligned}$$

where we've used the fact that double-frequency light at the output D is due to light in the circular fiber leaking out of junction A (i.e., $\mathcal{I} = \alpha^2\mathcal{I}_c = \alpha^2\mathcal{E}_c^2$). Thus, we have

$$\mathcal{I}_m = \frac{1}{4}I_0.$$

From the equality condition of the AM-GM inequality, we know that this maximum is achieved when

$$\begin{aligned} \alpha_m E_{c,m} &= 2\delta\mathcal{E}_{c,m}E_{c,m} \\ \alpha_m^2 &= 4\delta^2\mathcal{E}_{c,m}^2 \\ \alpha_m^4 &= 4\delta^2\alpha_m^2\mathcal{E}_{c,m}^2 = 4\delta^2\mathcal{I}_m = \delta^2 I_0 \\ \alpha_m &= \sqrt{\delta\sqrt{I_0}}. \end{aligned}$$

Now, the reader may notice that, as the amplitude $\sqrt{\alpha I_0}$ of the light leaking into the circular fiber at junction A adds onto the amplitude $\sqrt{1-\alpha}E_c$[8] of the existing wave inside the fiber, there seems to be extra energy $\left(\sqrt{\alpha I_0} + \sqrt{1-\alpha}E_c\right)^2 - \alpha I_0 - (1-\alpha)E_c^2 = 2\sqrt{\alpha(1-\alpha)I_0}E_c$ being created, which seemingly violates the law of conservation of energy. However, this can be explained by destructive interference[9] between the wave leaking out of the circular fiber at junction A (amplitude $\sqrt{\alpha}E_c$) and the existing wave in the straight fiber (amplitude $\sqrt{(1-\alpha)I_0}$): the energy loss here is $\alpha E_c^2 + (1-\alpha)I_0 - \left(\sqrt{(1-\alpha)I_0} - \sqrt{\alpha}E_c\right)^2 = 2\sqrt{\alpha(1-\alpha)I_0}E_c$, which cancels out with the extra energy in the circular fiber.

[7] $\mathcal{E}' = \sqrt{1-\alpha^2}\mathcal{E} \approx (1-\alpha^2/2)\mathcal{E}$, so $\Delta\mathcal{E} = -\alpha^2\mathcal{E}/2$.

[8] Here, let E_c refer to the exact amplitude of the main-frequency wave in the circular fiber right before reaching junction A.

[9] Note that the wave leaking into the circular fiber at junction A must be $\pi/2$ ahead/behind the input wave. This is because the waveform leaking into the circular fiber and the remaining waveform in the straight fiber add up to the input waveform, whereas their intensities add up to the input intensity. Similarly, the wave leaking out of the circular fiber at junction A is $\pi/2$ ahead/behind the wave in the circular fiber, hence constituting a total phase difference of π with the wave in the straight fiber.