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Problem No 5

Let κ denote the fraction of energy lost by the main-frequency wave while travelling over one semiarc inside the circular fibre. Figure 1 shows (real) amplitudes of the main frequency wave (written before coma) and the double-frequency wave (written after coma) in close neighbourhood of the couplings.

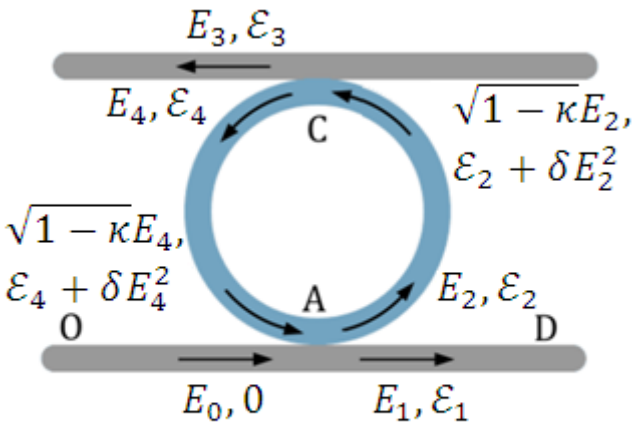

fig. 1

Energy carried by the waves is conserved in the circular fibre, so:

$$(1-\kappa)E_2^2 + (\mathcal{E}_2 + \delta E_2^2)^2 = E_2^2 + \mathcal{E}_2^2 \tag{1}$$

$$\kappa = 2\delta\mathcal{E}_2 + (\delta E_2)^2 \tag{2}$$

Using assumption $\delta\mathcal{E} \ll 1$, we get $\delta\mathcal{E}_2 \ll 1$ and $\delta(\mathcal{E}_2 + \delta E_2^2) = \delta\mathcal{E}_2 + (\delta E_2)^2 \ll 1$. It means that $\kappa \ll 1$. At the coupling C a fraction α of the main-frequency wave energy is given away, so $E_4 = \sqrt{1-\alpha}\sqrt{1-\kappa}E_2$. At the coupling A, a part of the laser wave is transmitted into the circular fibre. This part has amplitude $\sqrt{\alpha}E_0$. Its phase must be perfectly matched to the phase of E_2, because light in the circular fibre is just light from the laser circling there many times. Using the superposition principle we can write:

$$\sqrt{1-\alpha}\sqrt{1-\kappa}E_4 + \sqrt{\alpha}E_0 = E_2 \tag{3}$$

Expressing E_4 in terms of E_2, we can solve for E_2:

$$(1-\alpha)(1-\kappa)E_2 + \sqrt{\alpha}E_0 = E_2 \tag{4}$$

$$E_2 = \frac{\sqrt{\alpha}}{\alpha + \kappa - \alpha\kappa}E_0 \approx \frac{\sqrt{\alpha}}{\alpha + \kappa}E_0 \tag{5}$$

Let us track changes of the double-frequency wave energy along the circular fibre. On the right of the coupling A, we have $\mathcal{E}_2^2$. After travelling one semiarc, this energy increases by κE_2^2. Passing

through the coupling C decreases this energy by a factor of $1-\alpha^2$. Then it increases by κE_4^2 and finally decreases by a factor of $1-\alpha^2$ at the coupling A, to return to the initial value $\mathcal{E}_2^2$. Writing this as an equation, we get:

$$[(\mathcal{E}_2^2+\kappa E_2^2)(1-\alpha^2)+\kappa E_4^2](1-\alpha^2)=\mathcal{E}_2^2 \tag{6}$$

From this we can solve for $\mathcal{E}_2^2$:

$$\mathcal{E}_2^2=\frac{\kappa E_2^2(1-\alpha^2)^2+\kappa E_4^2(1-\alpha^2)}{(2-\alpha^2)\alpha^2}\approx\frac{\kappa E_2^2}{\alpha^2}=\frac{1}{\alpha}\frac{\kappa}{(\alpha+\kappa)^2}E_0^2 \tag{7}$$

Equations $\mathcal{E}_1^2=\alpha^2(\mathcal{E}_4+\delta E_4^2)^2$ and $\mathcal{E}_2^2=(1-\alpha^2)(\mathcal{E}_4+\delta E_4^2)^2$ yield:

$$\mathcal{E}_1^2=\frac{\alpha^2}{1-\alpha^2}\mathcal{E}_2^2\approx\frac{\alpha\kappa}{(\alpha+\kappa)^2}E_0^2 \tag{8}$$

Inequality of arithmetic and geometric means states that $\frac{\alpha+\kappa}{2}\geq\sqrt{\alpha\kappa}$. It is equivalent to an inequality $\frac{\alpha\kappa}{(\alpha+\kappa)^2}\leq\frac{1}{4}$. Thus we found an upper limit $\frac{1}{4}E_0^2$ for $\mathcal{E}_1^2$. It is achieved only for $\kappa=\alpha$. If we find such α_m that $\kappa=\alpha_m$, the problem will be solved. In order to do that, let us go back to the equation (2) and substitute for E_2 and $\mathcal{E}_2$ using equations (5) and (7):

$$\kappa=2\frac{\sqrt{\kappa/\alpha}}{\alpha+\kappa}\delta E_0+\frac{\alpha}{(\alpha+\kappa)^2}\delta^2E_0^2 \tag{9}$$

For $\alpha=\alpha_m$ we have $\kappa=\alpha_m$, so (9) is equivalent to:

$$\alpha_m=\sqrt{\delta E_0+\frac{1}{4}\delta^2E_0^2} \tag{10}$$

Term $\frac{1}{4}\delta^2E_0^2$ can be neglected, because $\delta E_0\ll\delta\mathcal{E}_2\ll 1$. Using $I_0=E_0^2$ we get finally:

$$\begin{aligned} I_m&=\frac{1}{4}I_0\\ \alpha_m&=(\delta^2I_0)^{1/4}\end{aligned} \tag{11}$$