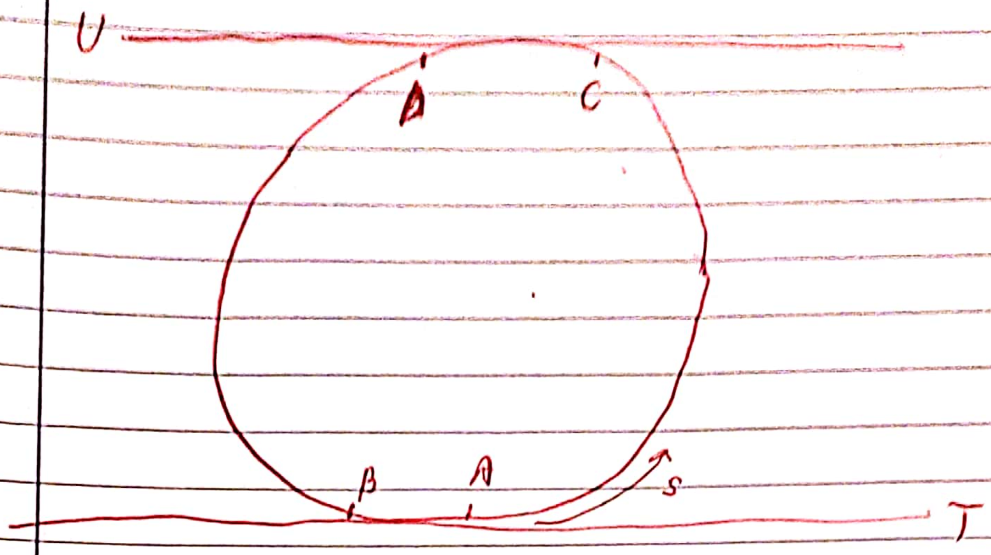
Physics Cup 2021 P5



let A, B near bottom coupling
C, D near top coupling

let distance along loop be S.

$$\therefore \begin{cases} \dfrac{dE(2\omega)}{dS} = +\dfrac{\delta}{\pi r} E^2(\omega) \\ \dfrac{dE(\omega)}{dS} = -\dfrac{\delta}{\pi r} E(\omega)E(2\omega) \end{cases}$$

(This is from $|E(2\omega)|^2 + |E(\omega)|^2 = \text{const.}$
$\Rightarrow 2E(2\omega)\dfrac{dE(2\omega)}{dS} + 2E(\omega)\dfrac{dE(\omega)}{dS} = 0$)

Let $E(2\omega, A)$ and $E(\omega, A)$ be respective amplitude at A

$$\therefore \frac{dE(2\omega)}{dS} = \frac{\delta}{\pi r}\left(E^2(2\omega, A) + E^2(\omega, A) - E^2(2\omega)\right)$$

Solving the ODE,

$$\frac{1}{\sqrt{E^2(2w,A)+E^2(w,A)}} \operatorname{arctan}\left(\frac{E(2w)}{\sqrt{E(2w,A)+E^2(w,A)}}\right) = \frac{\delta}{\pi r} s + C'$$

Set upper boundary at C

$$\therefore E(2w,C) = \sqrt{E^2(2w,A)+E^2(w,A)} \tanh\left(\delta\sqrt{E^2(2w,A)+E^2(w,A)} + \operatorname{arctanh}\left(\frac{E(2w,A)}{\sqrt{E^2(2w,A)+E^2(w,A)}}\right)\right)$$

let $A_0 = \sqrt{E^2(2w,A)+E^2(w,A)}$, and apply $A_0^2 = E^2(2w,C)+E^2(w,C)$

$$\therefore \begin{cases} E(2w,C) = A_0 \tanh\left(\delta A_0 + \operatorname{arctanh}\left(\frac{E(2w,A)}{A_0}\right)\right) \\ E(w,C) = A_0 \operatorname{sech}\left(\delta A_0 + \operatorname{arctanh}\left(\frac{E(2w,A)}{A_0}\right)\right) \end{cases}$$

Due to coupling to the upper straight fibre

$$E(2w,D) = \sqrt{1-\alpha^2}\, E(2w,C)$$
$$E(w,D) = \sqrt{1-\alpha}\, E(w,C)$$

let $D_0 = \sqrt{E^2(2w,D)+E^2(w,D)}$

Similarly we have

$$\begin{cases} E(2w,B) = D_0 \tanh\left(\delta D_0 + \operatorname{arctanh}\left(\frac{E(2w,D)}{D_0}\right)\right) \\ E(w,B) = D_0 \operatorname{sech}\left(\delta D_0 + \operatorname{arctanh}\left(\frac{E(2w,D)}{D_0}\right)\right) \end{cases}$$

Notice that intensities at B and at A are related.

Since light from input coupled into circular ring and light from B to A would interfere constructively ($2\pi r = n\lambda$)

$$\therefore \sqrt{1-\alpha}\, E(\omega, B) + \sqrt{I_0 \alpha} = E(\omega, A)$$

and $$E(2\omega, A) = \sqrt{1-\alpha^2}\, E(2\omega, B)$$

We can express intensities at B by $A \to C \to D \to B$, we now have relations $B \to A$

so we can have enough equations to solve for the system.

Call $\sqrt{1-\alpha}\,E(w,B) + \sqrt{I_0\alpha} = E(w,A)$ B.C. 1

$E(2w,A) = \sqrt{1-\alpha^2}\,E(2w,B)$ B.C. 2.

Expanding B.C. 1 :

$$-\alpha^2 E(2w,A) + \sqrt{1-\alpha^2}\left(E^2(w,A) - \alpha E^2(w,A) + \sqrt{1-\alpha^2}\,E^2(w,A)\right)\delta + O(\delta^2) = 0$$

$$\therefore\ E(2w,A) \approx \frac{\sqrt{1-\alpha^2}\,(1-\alpha+\sqrt{1-\alpha^2})\,E^2(w,A)}{\alpha^2}\delta$$

Expanding B.C. 2 and substitute $E(2w,A)$ in terms of $E(w,A)$

$$-\alpha E(w,A) + \sqrt{\alpha I_0} = E^3(w,A)\frac{(1-\alpha)\sqrt{1-\alpha^2}\,(1+\sqrt{1-\alpha^2})(1-\alpha+\sqrt{1-\alpha^2})\,\delta^2}{\alpha^2}$$

$$\therefore\ -\alpha E(w,A) + \sqrt{\alpha I_0} \approx \frac{4\delta^2}{\alpha^2}E^3(w,A) \qquad (\ddagger)$$

$$E(2w,T) = \frac{\sqrt{\alpha^2}}{\sqrt{1-\alpha^2}}E(2w,A)$$

$$= \frac{(1-\alpha+\sqrt{1-\alpha^2})\,E^2(w,A)\,\delta}{\alpha} \approx \frac{2\delta}{\alpha}E^2(w,A)$$

$$\therefore\ E^2(w,A) = \frac{\alpha E(2w,T)}{2\delta}$$

substitute to $(\ddagger)$

$$-\alpha\sqrt{\frac{\alpha E(2w,T)}{2\delta}} + \sqrt{\alpha I_0} = \frac{4\delta^2}{\alpha^2}\,\frac{\alpha E(2w,T)}{2\delta}\sqrt{\frac{\alpha E(w,T)}{2\delta}}$$

$$\sqrt{I_0} = \sqrt{E(2w,T)}\left(\frac{\alpha}{\sqrt{2\delta}} + \frac{E(2w,T)\sqrt{2\delta}}{\alpha}\right)$$

Using Arithmetic mean - Geometric mean inequality

$$\sqrt{I_0} = \sqrt{E(2w,T)}\left(\frac{\alpha}{\sqrt{2\delta}} + \frac{E(2w,T)\sqrt{2\delta}}{\alpha}\right)$$

$$\geq \sqrt{E(2w,T)}\; 2\sqrt{\frac{\alpha}{\sqrt{2\delta}}\frac{\sqrt{2\delta}}{\alpha}E(2w,T)}$$

$$\therefore \quad E(2w,T) \leq \frac{\sqrt{I_0}}{2}$$

$$I_m = E^2(2w,T)_{max} = \frac{I_0}{4}$$

Equality holds when $\frac{\alpha_m}{\sqrt{2\delta}} = \frac{E(2w,T)\sqrt{2\delta}}{\alpha_m}$

$$\therefore \quad \alpha_m^2 = 2\delta E(2w,T) = 2\delta\cdot\frac{\sqrt{I_0}}{2} = \delta\sqrt{I_0}$$

$$\therefore \quad \alpha_m = \sqrt{\delta\sqrt{I_0}}$$

final answer:

$\therefore$

$$I_m = \frac{I_0}{4}$$

$$\alpha_m = \sqrt{\delta\sqrt{I_0}}$$

A simpler but more hand-wavy approach

Because $\delta E^2 \ll 1$
so $E(2\omega)$ and $E(\omega)$ does not change much in the ring
they stabilize at high values, only marginal change.

Conserve total output energy:

$$2\alpha^2 E^2(2\omega) + \alpha E^2(\omega) + (\sqrt{\alpha}E(\omega) - \sqrt{1-\alpha^2}\sqrt{I_0})^2 = I_0$$

since $\alpha \ll 1$

$$2\alpha^2 E^2(2\omega) + \alpha E^2(\omega) + (\sqrt{\alpha}E(\omega) - \sqrt{I_0})^2 = I_0 \quad (1)$$

Energy output by second harmonic is essentially lost from main-wave

$$\Delta \text{Energy} = 2\alpha^2 E^2(2\omega) = 2E(2\omega)\cdot(2\delta E^2(\omega))$$

$$\therefore \frac{E^2(\omega)}{E(2\omega)} = \frac{\alpha^2}{2\delta} \quad (2)$$

from (1):

$$\alpha^2 E^2(2\omega) = \frac{I_0}{2} - \frac{\alpha E^2(\omega) + (-\sqrt{\alpha}E(\omega) + \sqrt{I_0})^2}{2}$$

$$\sqrt{\alpha}E(\omega) > 0, \quad \sqrt{I_0} - \sqrt{\alpha}E(\omega) > 0 \text{ (energy consent)}$$

using AM-QM inequality

$$\alpha^2 E^2(2\omega) \leq \frac{I_0}{2} - \left(\frac{\sqrt{\alpha}E(\omega) - \sqrt{\alpha}E(\omega) + \sqrt{I_0}}{2}\right)^2$$

$$\alpha^2 E^2(2\omega) \leq \frac{I_0}{2} - \frac{I_0}{4} = \frac{I_0}{4}$$

$$\therefore I_M = \frac{I_0}{4} \quad \text{when } E(\omega) = \frac{\sqrt{I_0}}{2\sqrt{\alpha}}$$

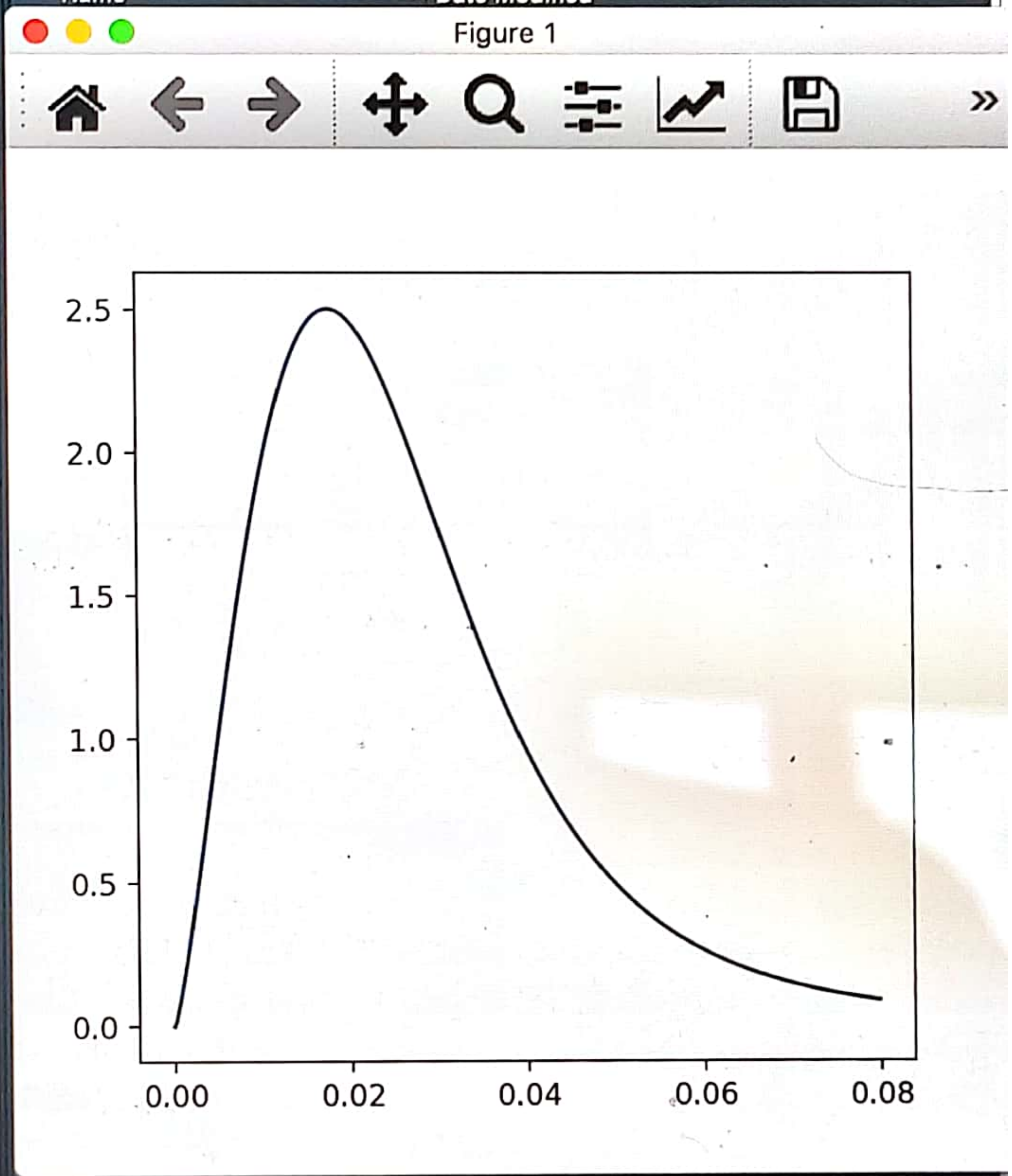
back to (2):

$$\frac{\frac{I_0}{4\alpha}}{\frac{\sqrt{I_0}}{2\alpha}} = \frac{\alpha_m^2}{2\delta} \qquad \alpha_m^2 = \delta\sqrt{I_0}$$

$$\therefore \alpha_m = \sqrt{\delta\sqrt{I_0}}$$

$$\therefore \boxed{\begin{aligned} \alpha_M &= \sqrt{\delta\sqrt{I_0}} \\ I_M &= \frac{I_0}{4} \end{aligned}}$$

```
    d2=np.sqrt(1-a**2)*c2
    d1=np.sqrt(1-a)*c1

    b2=np.sqrt(d1**2+d2**2)*np.tanh(S*np.sqrt(d1**2
    b1=np.sqrt(d1**2+d2**2)/np.cosh(S*np.sqrt(d1**2

    newa2=b2*np.sqrt(1-a**2)
    newa1=np.sqrt(1-a)*b1+np.sqrt(J*a)



    return newa1,newa2




a1=np.sqrt(J)
a2=0.0

a=0.0178



def I(a,sa1,sa2):
    crit1,crit2=aloop(sa1,sa2,a)



    while abs(crit1-sa1)+abs(crit2-sa2)>0.00001:
        sa1=abs((sa1+crit1)/2)
        sa2=abs((sa2+crit2)/2)
        crit1,crit2=aloop(sa1,sa2,a)

    return (a**2*crit2**2/(1-a**2),crit1,crit2)


N=100

x=[""]*N
y=[""]*N
for i in range(N):
    x[i]=i/N*0.08
    (y[i],a1,a2)=I(x[i],a1,a2)

matplotlib.pyplot.plot(x,y)
```



In [162]: plot[x,y]

IPython console History