Physics Cup 2023 – Problem 1

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A thin long wire is made of a material that undergoes a phase transition so that its resistivity takes one of the two values, ρ_1 is its temperature is smaller than T_c, and $\rho_2 = 2\rho_1$ if the temperature is larger than T_c.

Assume the following:

- The heat flux per unit length of the wire to the ambient medium is $W = \alpha(T - T_0)$ where α is a constant, T denotes the temperature of the wire, and T_0 is the ambient temperature ($T_0 < T_c$);
- The wire is so thin that the thermal flux along the wire can be almost everywhere neglected, and the characteristic thermalization time is much shorter than the time during which the voltage is changed;
- While the coefficient α and cross-sectional area A are almost constant along the wire, due to imperfections, there are tiny variations;
- The length of the wire $L \gg \sqrt{A}$.

The voltage V applied to the ends of the wire is increased very slowly at a constant rate (i.e. linearly in time), from zero until the whole wire has undergone a phase transition, and then reduced back to zero, at the same constant rate. Sketch how the total power P dissipated in the wire depends on time t.

Your sketch should show power and voltage values at any point where either P or its time derivative is discontinuous, expressed in terms of $P_0 = \alpha L(T_c - T_0)$ and $V_0 = L\sqrt{\frac{\alpha\rho_1(T_c - T_0)}{A}}$. The solution is considered to be correct only if all these points have correct values, and all segments of the graph are qualitatively correct (e.g. a convex curve should not be drawn as a straight line).

1 Step By Step

1.1 Two Temperatures

As there is almost no heat current through the wire, almost every wire infinitesimal is in thermal equilibrium with the environment. Because the current is constant through the sample, that means all the ρ_2 regimes have the same temperature – same I and same resistivity mean same Joule heating – hence same temperature. Same is true for all ρ_1 regimes of course.

Note that these temperatures are **different** – parts with higher temperature have higher resistivity, hence more heat is dissipated in them. Following Matthew effect: "For him who has will more be given, and he will have abundance; but from him who has not, even what he has will be taken away." (Matthew 13:12). The wire will have two temperatures, T_h for ρ_2 and T_l for ρ_1.

1.2 First Temperatures Condition

Suppose a steady state occurs where a segment of length ℓ is in the high temperature, and the rest $(L - \ell)$ is in the low temperature.

The overall resistance is $\frac{\rho_2\ell}{A} + \frac{\rho_1(L-\ell)}{A} = \frac{\rho_1 L}{A} + \frac{\rho_1}{A}\ell \equiv R_1 + r\ell = R_1\left(1 + \frac{\ell}{L}\right)$ and the current is $\frac{V}{R_1 + r\ell}$. We will use R_1, r from now on.

A unit length dL in phase 1 will experience voltage of $dV = I\frac{\rho_1 dL}{A} = \frac{V}{R_1\left(1+\frac{\ell}{L}\right)}\frac{R_1 dL}{L} = V\frac{dL}{L+\ell}$, hence omits heat of $IdV = V\frac{dL}{L+\ell}\frac{V}{R\left(1+\frac{\ell}{L}\right)} = \frac{V^2}{R}\frac{LdL}{(L+\ell)^2}$. Was it in phase 2, the voltage was $I\frac{2\rho_1 dL}{A} = 2V\frac{dL}{L+\ell}$. Heat would've been $\frac{2V^2 LdL}{R_1(L+\ell)^2}$. From this we directly see that $T_h - T_0 = \frac{2V^2 L}{\alpha R_1(L+\ell)^2}$ while $T_l - T_0 = \frac{V^2 L}{\alpha R_1(L+\ell)^2}$. We note that $(T_h - T_0) = 2\left(T_l - T_0\right)$.

1.3 Second Temperatures Condition

Look on a "marginal" piece of wire dL, the one that isn't sure wether it should be in ρ_1 or in ρ_2. Such infinitesimal piece must be in temperature T_c, with neighbors in both phases; hence, for the heat current to work and not accumulate, we get our second condition – $T_h - T_c = T_c - T_h$.

Combined with the previous condition, we can extract the temperatures – independently of ℓ! We have $T_l = T_0 + \frac{2}{3}\left(T_c - T_0\right) = \frac{2}{3}T_c + \frac{1}{3}T_0$ and $T_h = T_0 + \frac{4}{3}\left(T_c - T_0\right) = \frac{4}{3}T_c - \frac{1}{3}T_0$.

1.4 Finding ℓ

We now find the single allowed steady state. Note that larger ℓ leads to higher **outflux** (more hot part) and lower **influx** (higher resistance, lower heat emission), hence there is a single solution given V. We can actually just extract it from the expressions of T_h and T_l from the second section. $\frac{V^2 L}{\alpha R_1(L+\ell)^2} = \frac{2}{3}\left(T_c - T_0\right)$ gives $L+\ell = \sqrt{\frac{3}{2}\frac{L}{\alpha R_1(T_c - T_0)}}V$. We get ℓ that's linear in V! The minimal value is $V_{min} = \sqrt{\frac{2}{3}\alpha R_1\left(T_c - T_0\right)L}$, and the maximal is $V_{max} = \sqrt{\frac{8}{3}\alpha R_1\left(T_c - T_0\right)L}$.

2 Overall Result

There is one more complication to consider. The influx-outflux argument gives only one solution in which there is an equilibrium of two phases. Carefully heating can lead us to a "metastable" state of only R_1 all the way up to T_c. This will happen in voltage $V_0 \equiv \sqrt{\alpha R_1\left(T_c - T_0\right)L}$. The wire will heat in rate of $\frac{V_0^2}{R_1} = \alpha\left(T_c - T_0\right)L = P_0$, of course. Then we will "jump" to a state with finite ℓ, hence with a qualitatively **lower** heat emission – $L+\ell = \sqrt{\frac{3}{2}}L$, hence $\ell = \sqrt{\frac{3}{2}} - 1$. The emission is $\alpha\left(T_l - T_0\right)\left(L - \ell\right) + \alpha\left(T_h - T_0\right)\ell = \left(1 - \frac{\ell}{L}\right)\frac{2}{3}P_0 + \frac{\ell}{L}\frac{4}{3}P_0 = \frac{2}{3}\left(1 + \frac{\ell}{L}\right)P_0 = \sqrt{\frac{2}{3}}P_0$. We will go all the way up to V_{max}, and then will find ourself in a state of a single, high temperature – now with $P = \frac{4}{3}P_0$.

Cooling, We will go down to $\sqrt{2}V_0$, heating in $\frac{2V_0^2}{R_2} = P_0$, and then experience another jump – $L+\ell = \sqrt{3}L$, hence we'll suddenly get $P = \frac{2}{\sqrt{3}}P_0$, as a large portion suddenly became more conducting. This will cool down all the way to the single low temperature with $\frac{2}{3}P_0$, and then of course to zero.

In the intermediate state we note that the heat emission is $\alpha\left(L+\ell\right)\left(T_l - T_0\right) = \frac{2}{3}\left(1 + \frac{\ell}{L}\right)P_0$. But as $L+\ell$ is linear in V, we get that P is linear in V as well – $P = \frac{2}{3}\frac{L+\ell}{L}P_0 = \sqrt{\frac{2}{3}}\sqrt{\frac{1}{\alpha R_1 L(T_c - T_0)}}VP_0 = \sqrt{\frac{2}{3}}\frac{V}{V_0}P_0$

A full description, therefore:

- Simple voltage raising – $P \propto V^2$ up to $V = V_0$ and $P = P_0$.
- A discrete jump to $P = \sqrt{\frac{2}{3}}P_0$.
- A linear rise up to $P = \frac{4}{3}P_0$, $V = \sqrt{\frac{8}{3}}V_0$.
- *[A possible quadratic rise further, but the cooling stops]*
- A quadratic decay, $P \propto V^2$ down to $T = T_c$ with $V = \sqrt{2}V_0$ and $P = P_0$.

- A discrete jump to a $\frac{2}{\sqrt{3}}P_0$ in the same voltage.
- A linear decay down to $P = \frac{2}{3}P_0$ in $V = \sqrt{\frac{2}{3}}V_0$.
- A quadratic decay down with $P \propto V^2$.