Problem No 2 - Gil Ronen

1 Phase 1 dynamics

The wire starts at equilibrium with the environment, at temperature $T_0 < T_c$. When the voltage starts being applied, the output power goes by $P = \frac{V^2}{R_1}$ where $R_1 = \rho_1 \frac{L}{A}$. This situation continues to hold until

$$P = P_0 = \alpha L \left(T_c - T_0\right)$$

$$V = V_0 = \sqrt{P_0 R_1} = L\sqrt{\alpha \rho_1 \left(T_c - T_0\right)/A}$$

At this point, all of the wire is at a temperature T_c. The wire segment with the smallest α (since there are imperfections in the wire) is heated more, and transforms to the second phase.

2 Mixed phase dynamics

Now we study the dynamics when there are two phases in the wire. We denote by x the portion of the wire in the second phase and the each phase temperature by T_1, T_2 respectively[1].

2.1 Temperatures of the phases

1. The temperatures must satisfy $T_1 < T_c < T_2$.
2. Second, in a stationary configuration, **the average temperature must equal** T_c or else the boundry between the phases would be "pushed" to some direction.
3. Third, given a current I flowing through the wire, the power dissipated per unit length in each of the phases is:

$$\frac{I^2 \rho_i}{A}$$

[1] The temperature of a given phase is the same everywhere since the current is constant along the wire and the temperature is determined by the equilibrium between heat flux to the ambient medium and the heat generated by the flow of electricity (which is proportional to the resistivity of the phase).

which must equal the heat flux per unit length which is:

$$\alpha\left(T_i - T_0\right)$$

so

$$\beta = \frac{I^2}{\alpha A} = \frac{\Delta T_i}{\rho_i}$$

is the same constant for each of the phases. Thus:

$$\Delta T_2 = \Delta T_1 \frac{\rho_2}{\rho_1} = 2\Delta T_1$$

With remarks 2 and 3, we arrive at the conclusion:

$$\boxed{\Delta T_1 = \frac{2}{3}\Delta T_c,\ \Delta T_2 = \frac{4}{3}\Delta T_c}$$

$$\beta = \frac{2}{3}\frac{\Delta T_c}{\rho_1}$$

2.2 Power dissipation

The total resistance of the wire is:

$$R\left(x\right) = \frac{L}{A}\left(\rho_1\left(1-x\right) + \rho_2 x\right) = \frac{L}{A}\rho_1\left(1+x\right)$$

and the current flowing in the wire satisfies $I = V/R$. so two equations must hold:

$$\frac{V^2\rho_i}{AR\left(x\right)^2} = \alpha\Delta T_i$$

specializing for one of them (it doesn't matter which):

$$R\left(x\right) = \frac{V}{\sqrt{\alpha\beta A}}$$

So the total power dissipated in the wire is:

$$P\left(x\right) = \frac{V^2}{R\left(x\right)} = V\sqrt{\frac{2}{3}\alpha\frac{\Delta T_c}{\rho_1}A} = \sqrt{\frac{2}{3}}P_0\frac{V}{V_0}$$

and x can be found with:

$$x = \sqrt{\frac{3}{2}}\frac{V}{L}\sqrt{\frac{A}{\alpha\rho_1\Delta T_c}} - 1 = \sqrt{\frac{3}{2}}\frac{V}{V_0} - 1$$

and the boundries:

$$0 < x < 1$$

correspond to:

$$\sqrt{\frac{2}{3}}V_0 < V < 2\sqrt{\frac{2}{3}}V_0$$

3 Overall dynamics

1. From 0 to V_0 the power rises quadratically with $V^2/R(0)$. At V_0, where the power is P_0, the wire enters the mixed stage, and the power falls to $\sqrt{\frac{2}{3}}P_0$ and $x = \sqrt{\frac{3}{2}} - 1$. At this point the function of the power is discontinuous.

2. Then, as the voltage rises, the power rises linearly as $\sqrt{\frac{2}{3}}P_0\frac{V}{V_0}$ until $V = 2\sqrt{\frac{2}{3}}V_0$ and $x = 1$, meaning the whole wire is now in phase 2. The power at this point is $\frac{4}{3}P_0$ and the temperature of the wire is $T_2 = T_0 + \frac{4}{3}\Delta T_c$.

3. Then, when the voltage lowers, the wire doesn't fall right back to the mixed phase and stays metastable until the temperature of the wire drops below T_c. Until this point, the voltage dependence of the power is $P = V^2/R(1) = V^2/2R(0)$. The phase change happens at $V = \sqrt{2}V_0$, where the power is $P = P_0$. Once again the function of the power itself is discontinuous.

4. Right after the phase change, the power jumps to $\frac{2}{\sqrt{3}}P_0$, and $x = \sqrt{3} - 1$. The voltages then lowers to $\sqrt{\frac{2}{3}}V_0$, at this point the power is $\frac{2}{3}P_0$ and the whole wire is in phase 1. Now the derivative of the power is discontinuous but the power itself is continuous.

5. From $V = \sqrt{\frac{2}{3}}V_0$ to 0, the power drops quadratically as $V^2/R(0)$.

4 Graph

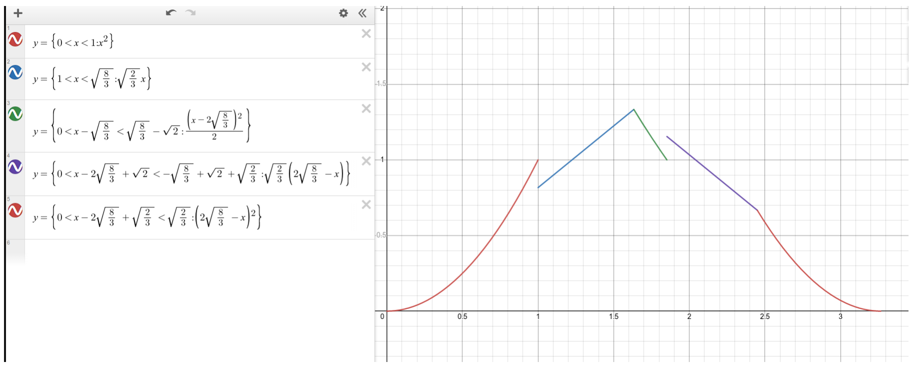

Figure 1: