Physics Cup 2023
Problem 2: Phase transitions in a wire

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1 Interpretation and basic idea of the solution

We assume that the voltage is being increased quasi-statically. This will guarantee that thermal equilibrium has been achieved at all values of voltage applied. Initially let's assume that the entire wire was at room temperature T_0. Neglecting heat conduction along the wire, the wire gets heated to just below T_c at voltage $V = V_0$ (say).
The equilibrium condition is:

$$\frac{A{V_0}^2}{\rho_1 L} - \alpha L(T_c - T_0) = 0$$

$$\Rightarrow V = V_0 = L\sqrt{\alpha\rho_1(T_c - T_0)/A}$$

At this voltage, power generated by wire P_0 is,

$$P_0 = \frac{A{V_0}^2}{\rho_1 L} = \alpha L(T_c - T_0)$$

Before reaching this voltage, the total power generated by the wire (equal to the power dissipated as heat) be P.

$$P = \frac{V^2}{R} = \frac{AV^2}{\rho_1 L} = P_0\left(\frac{V}{V_0}\right)^2$$

Since, there are manufacturing defects in the cross section of the wire, there would be points where the cross section is less than A, and there would also be a minimum value. At such points, phase transitions will begin as temperature locally exceeds T_c.
Let f fraction of the length has undergone phase transition to ρ_2. The resistance of the whole wire will be R.

$$R = \frac{f\rho_2 L}{A} + \frac{(1-f)\rho_1 L}{A} = \frac{(1+f)\rho_1 L}{A}$$

Power generated by wire per unit length of wire (at a point with resistivity ρ) be p

$$p = \frac{I^2\rho}{A} = \frac{\rho A V^2}{(1+f)^2\rho_1^2 L^2}$$

Now to calculate what is the power output of the wire during the transition process, we need to know what is the fraction f at any voltage applied. That in turn depends on the temperature distribution of the wire (how much part of wire is above T_c).
We will make use of the general 1-D heat conduction equation for this purpose.

2 Finding temperature distribution of the wire

At any point x on the wire, we can write:

$$k\frac{\partial^2 T}{\partial x^2} + \dot{e_{gen}} - \dot{e_{rem}} = dc\frac{\partial T}{\partial t}$$

Where, k is thermal conductivity; d is density; c is specific heat capacity; $\dot{e_{gen}}$ is power generated per unit length. $\dot{e_{rem}}$ is heat removed per unit time per unit length.
As we are working with equilibrium condition, temperature won't be time dependent. So, for this problem:

$$k\frac{d^2T}{dx^2} + \frac{\rho AV^2}{(1+f)^2\rho_1^2L^2} - \alpha(T - T_0) = 0 \tag{1}$$

We do note that this value of thermal conductivity is so small that it can be neglected *almost everywhere*. There will be a finite number of discrete points (a set of measure zero) where this can't be neglected, as we shall see.

2.1 In an interior point, away from phase boundaries

We approximate k to be 0 since heat current is negligible. Therefore we use equation (1) with k set to 0.

$$\alpha(T_c - T_0) = \frac{\rho AV^2}{(1+f)^2\rho_1^2L^2}$$

$$\Rightarrow T = T_0 + \frac{\rho AV^2}{(1+f)^2\alpha\rho_1^2L^2}$$

Thus, the temperature at any interior point within a phase of resistivity ρ is constant and are:
For phase ρ_1

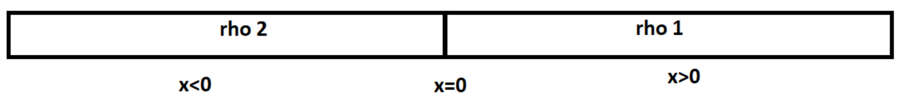
$$T = T_0 + \frac{AV^2}{(1+f)^2\alpha\rho_1L^2}$$

For phase ρ_2

$$T = T_0 + \frac{2AV^2}{(1+f)^2\alpha\rho_1L^2}$$

2.2 At a phase boundary

There will be a finite number of such phase boundaries since nucleation will begin only at finite points (points where phase change will begin). At such boundaries we can't ignore the conduction term in our equation (1) because there exists a finite temperature difference across a very short distance, i.e. near infinite thermal gradient. The plan is to do our calculation by assuming a finite k and then taking a limit to 0.
Let such a phase boundary be as shown in the figure:



The transition from one phase to other is occurring at the phase boundary, and this transition happens at T_c. **Thus, the temperature of the wire at the boundary $x = 0$, must be T_c.**
For, $x > 0$:

$$k\frac{d^2T}{dx^2} + p - \alpha(T - T_0) = 0$$

Let $T' = T - T_0$, we solve this differential equation to get:

$$T = T' + T_0 = T_0 + \frac{p}{\alpha} + T_ae^{-\lambda x} + T_be^{\lambda x}$$

$$\lambda = \sqrt{\frac{\alpha}{k}}$$

Putting boundary condition, that T is almost constant at interior points, we can say T_b must be zero (for finite temperatures). Also we have argued that at $x = 0$, $T = T_c$, therefore,

$$T_c - T_0 = \frac{p}{\alpha} + T_a$$

Now, for $x < 0$ we have:

$$k\frac{d^2T}{dx^2} + 2p - \alpha(T - T_0) = 0$$

We solve to get,

$$T = T' + T_0 = T_0 + \frac{2p}{\alpha} + T_d e^{-\lambda x} + T_b e^{\lambda x}$$

$$\lambda = \sqrt{\frac{\alpha}{k}}$$

T_d must be zero (for finite temperatures at $x < 0$). Also since at $x = 0$, $T = T_c$,

$$T_c - T_0 = \frac{2p}{\alpha} + T_b$$

We have another physical condition to exploit: **that is the outward heat flux from the ρ_2 phase along the conductor must be equal to the heat flux into ρ_1 phase.**. In other words, the gradient of temperature on both side of zero must be continuous.

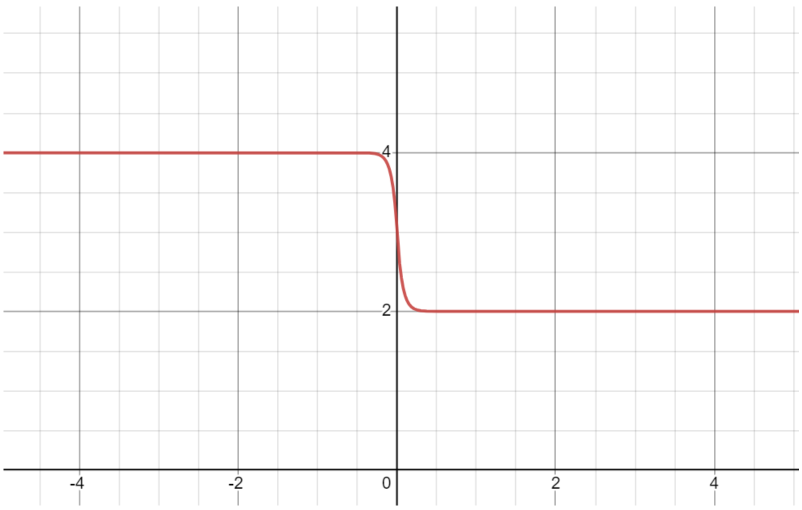


Figure 1: A plot of this temperature distribution at phase boundary. In this limit $k \to 0$, this will limit the transition zone to a point of measure 0

For $x < 0$, $\frac{dT}{dx} = \lambda T_b$ and for $x > 0$, $\frac{dT}{dx} = -\lambda T_a$. Since both of them are equal we have $T_a + T_b = 0$.
Using these boundary relations, we can eliminate T_a and T_b to get,

$$T_c - T_0 = \frac{3}{2}p = \frac{3}{2}\frac{AV^2}{(1+f)^2\alpha\rho_1 L^2}$$

Simplifying this result to have $1 + f$ separated,

$$1 + f = \sqrt{\frac{3}{2}\frac{V}{V_0}} \tag{2}$$

3 Plot of the heating process from ρ_1 to ρ_2

We see that as soon as the phase change begins, a fraction $f = \sqrt{1.5} - 1 \approx 0.225$ of the wire converts instantaneously, as can be seen from equation (2).
Once this process has started, the subsequent power dissipation is to be calculated:

$$P = \frac{V^2}{R} = \frac{AV^2}{(1+f)\rho_1 L}$$

$$\Rightarrow P = \sqrt{\frac{2}{3}} \frac{AV_0^2}{\rho_1 L} \frac{V}{V_0} = \sqrt{\frac{2}{3}} P_0 \frac{V}{V_0}$$

This will continue till the entire wire has converted to phase ρ_2. After complete transformation, the power dissipated is given by:

$$P = \frac{V^2}{R} = \frac{AV^2}{\rho_2 L} = \frac{1}{2} P_0 \left(\frac{V}{V_0}\right)^2$$

To summarise the entire graph of heating process is shown with all the critical points marked.

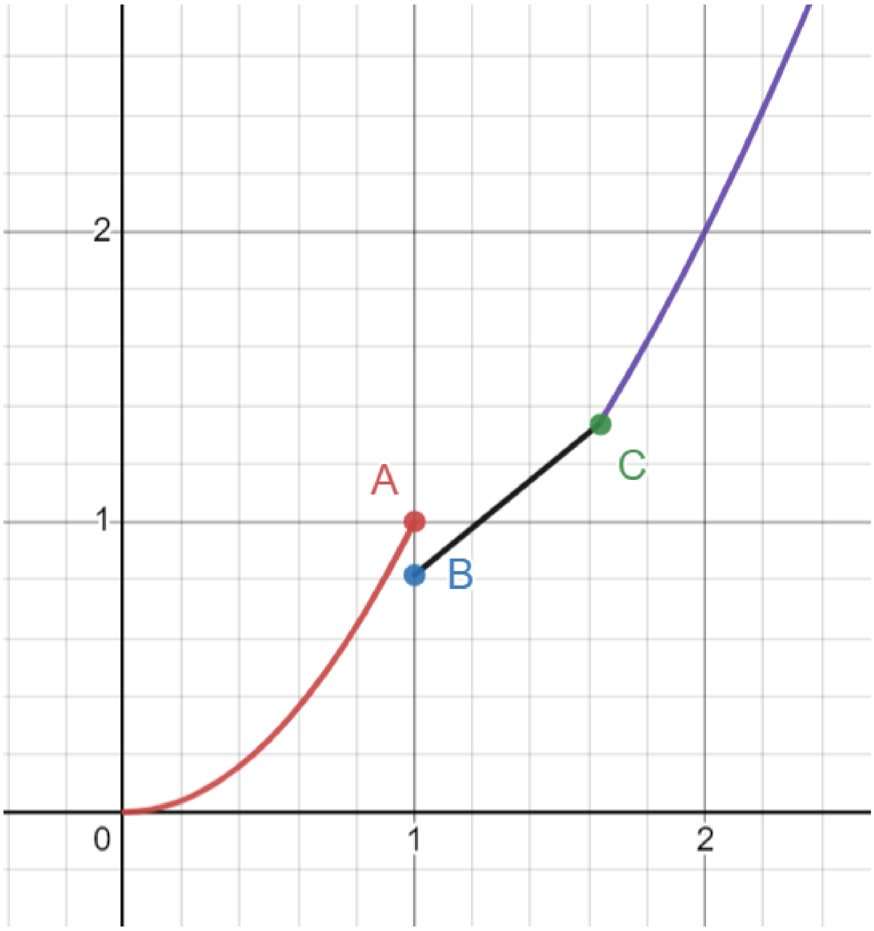


Figure 2: Graph of power dissipation versus voltage, during **heating**

The points A, B and C are (in units of V_0 and P_0): **A:**$(1, 1)$**, B:**$(1, \sqrt{\frac{2}{3}})$**, C:**$(2\sqrt{\frac{2}{3}}, \frac{4}{3})$.
The red curve is $P = P_0 \left(\frac{V}{V_0}\right)^2$. The black straight line is $P = \sqrt{\frac{2}{3}} P_0 \frac{V}{V_0}$. And the purple curve is $P = \frac{1}{2} P_0 \left(\frac{V}{V_0}\right)^2$

4 Plot of the cooling process from ρ_2 to ρ_1

Now we try to determine the plot when the wire completely in phase ρ_2 is cooled down to phase ρ_1. We again assume that initially the wire had a uniform temperature $T_H > T_c$.
No phase change will happen until the entire wire is just above T_c. After that, at those points where cross-section is more than A, i.e. resistance is a bit less, the phase change will begin. The voltage at which this will happen is $V = V_1$, say.

$$V_1 = L\sqrt{\alpha\rho_2(T_c - T_0)/A} = \sqrt{2}V_0$$

Once the phase transition begins, the fraction of phase ρ_2 is given by equation (2).

$$1 + f = \sqrt{\frac{3}{2}}\frac{V}{V_0}$$

At the very beginning, phase ρ_2 is instantaneously reduced to $f = 0.732$, after that the transformation continues till $f = 0$, or $V = \sqrt{\frac{2}{3}}V_0$
The relation between power and voltage during phase transition will be same as what we have calculated in the heating part.

$$P = \sqrt{\frac{2}{3}}P_0\frac{V}{V_0}$$

After the transformation is complete, the power versus voltage relation is also known.

$$P = P_0\left(\frac{V}{V_0}\right)^2$$

This has been summarised in this attached cooling curve.

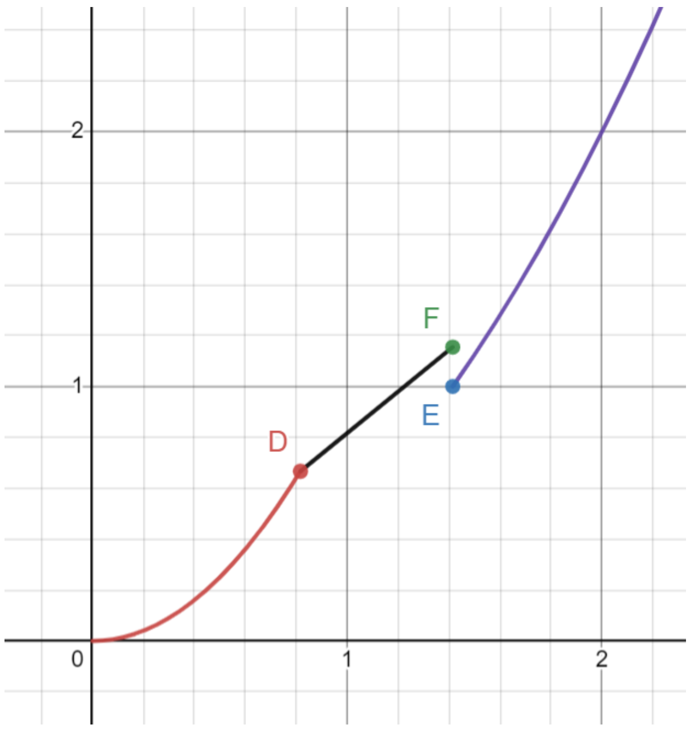


Figure 3: Graph of power dissipation versus voltage, during **cooling**

The points D, E and F are (in units of V_0 and P_0): **A:**$\left(\sqrt{\frac{2}{3}}, \frac{2}{3}\right)$**, E:**$(\sqrt{2}, 1)$**, F:**$\left(\sqrt{2}, \frac{2}{\sqrt{3}}\right)$.
The red curve is $P = P_0\left(\frac{V}{V_0}\right)^2$. The black straight line is $P = \sqrt{\frac{2}{3}}P_0\frac{V}{V_0}$. And the purple curve is $P = \frac{1}{2}P_0\left(\frac{V}{V_0}\right)^2$