Physics Cup 2023 - Problem 2

(taking into account longitudinal heat transfer)

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Contents

1 Introduction

This solution is organised as follows: first, I will roughly analyze how the wire responds to various values of the input voltage. I will particularly look at what happens during the phase change. Afterwards, using what was found about the general behaviour of the wire, I will consider how it reacts to the voltage input described in the problem text.

2 General considerations

Let I be the current that flows through the wire. The values of α and A can vary slightly between different parts of the wire; we will denote α_l and A_l the local values of the two parameters. However, I is constant in the whole wire, as a consequence of the conservation of charge. Hence, let us analyze the thermal equilibrium of this portion of the wire. Denoting by ρ_l the local value of the resistivity, we have that, for a short portion of the wire of length dl, that[1]

$$dP = \alpha_l dl (T_l - T_0) = I^2 \frac{\rho_l dl}{A_l} \Longrightarrow T_l = T_0 + \frac{\rho_l I^2}{\alpha_l A_l}. \tag{1}$$

As the voltage (and hence I) increases from 0, the resistivity of the wire segment will be ρ_1. When the current becomes large enough, the part of the wire we analyzed above (the one for which the l indices are defined) will undergo the phase change. This happens when $T_l = T_c$, or, in other words, when

$$I = I_{\uparrow,l} = \sqrt{\alpha_l A_l \frac{T_c - T_0}{\rho_1}}. \tag{2}$$

Since $\alpha_l \approx \alpha$ and $A_l \approx A$ everywhere along the wire, this corresponds approximately to

$$I_\uparrow = \sqrt{\alpha A \frac{T_c - T_0}{\rho_1}}, \tag{3}$$

which is hence the current at which the wire will begin the phase transition.

Similarly, if the current is large enough for the whole wire to be in the high-resistivity state and we slowly lower the voltage, there will be a certain current at which the short wire

[1] In this statement, I assume that neighboring segments of the wire do not exchange heat, which is true under the problem conditions.

segment will revert to the low-resistance state. This happens, once again, when $T_l = T_c$. Now $\rho_l = 2\rho_1$, so that this value of the current is given by

$$I = I_{\downarrow,l} = \sqrt{\alpha_l A_l \frac{T_c - T_0}{2\rho_1}}, \tag{4}$$

which means that the approximate current at which the wire will begin the phase transition is

$$I_\downarrow = \sqrt{\alpha A \frac{T_c - T_0}{2\rho_1}} = \frac{I_\uparrow}{\sqrt{2}}. \tag{5}$$

3 The phase transitions

We have found at what values of the intensity the phase transitions will begin. Let us analyze each of the phase transitions in more detail, taking into account what the effect of the weak longitudinal heat transfer is. This time, we will assume that the conditions under which these situations occur are the same as in the problem text – we will consider that the *voltage* increases very slowly and is approximately constant at any point in time.

3.1 The upwards phase transition

Let us first look at the transition from the low-resistivity phase to the high-resistivity phase. As the voltage increases from 0, the resistivity of the whole wire is ρ_1. The wire will remain in this state until the current reaches $I_\uparrow$. This will occur when the voltage is

$$V = I_\uparrow \frac{\rho_1 L}{A} = V_0. \tag{6}$$

Afterwards, a certain small part of the wire will change state, its resistivity increasing to $2\rho_1$. Since this part of the wire is small, the overall current will remain approximately constant; however, the increase in resistivity will make the small wire segment heat up significantly more. Due to the weak, but present, longitudinal heat transfer in the wire, this will also make the neighboring segments heat up. Since these were also very near T_c, this will make them change state too, and the process goes on, with more and more of the wire changing state in a very small amount of time, until heat transfer equilibrium is achieved.[2] Since

[2]This will actually occur, because the change of the state of the regions leads to an increase in the overall

heat transfer is linear in temperature, equilibrium will be reached when the mean of the temperatures of the two parts of the wire will be equal to T_c — if this mean were higher, the boundary of the high-resistance state would tend to broaden, and if it were lower, this boundary would tend to shrink.

Let I_t be the current at which equilibrum is achieved. The temperatures of the two sections are

$$\begin{aligned} T_1 &= T_0 + \frac{\rho_1 I_t^2}{\alpha A} = T_0 + (T_c - T_0)\left(\frac{I_t}{I_\uparrow}\right)^2; \\ T_2 &= T_0 + \frac{\rho_2 I_t^2}{\alpha A} = T_0 + 2(T_c - T_0)\left(\frac{I_t}{I_\uparrow}\right)^2; \end{aligned} \tag{7}$$

so that the above condition becomes

$$\frac{T_1 + T_2}{2} = T_c \Longrightarrow I_t = \sqrt{\frac{2}{3}} I_\uparrow. \tag{8}$$

As the voltage continues to increase, more and more of the wire will change state, while the current will stay constant at I_t. This process ends when the whole wire has changed state, i.e.

$$V = I_t \frac{\rho_2 L}{A} = \sqrt{\frac{8}{3}} V_0. \tag{9}$$

3.2 The downwards phase transition

The downwards transition occurs similarly. As the voltage decreases, the whole wire stays in the high-resistance state until $I = I_\downarrow$. Afterwards, for similar reasons (this time caused by the cooling of regions adjacent to the one that first changes state), the current will decrease to I_t and stay at this value until the whole wire has changed state. This process begins when

$$V = I_\downarrow \frac{\rho_2 L}{A} = V_0 \sqrt{2}, \tag{10}$$

and ends when the whole wire is in the ρ_1 state, i.e.

$$V = I_t \frac{\rho_1 L}{A} = \sqrt{\frac{2}{3}} V_0. \tag{11}$$

resistance of the wire, which lowers the current. After enough of the wire has changed state, the current will reach this value — it is guaranteed that it actually will, because, as it will be seen later, I_t lies between $I_\downarrow$ and $I_\uparrow$, and if the whole wire were to change state, the current in the wire would be less than $I_\downarrow$, which proves that only a part of the wire undergoes the phase transition. A similar argument can be applied for the downwards phase transition.

4 Calculation of the power

Now, we are ready to find how the power depends on voltage over the course of the voltage variation described in the problem text. Since voltage depends linearly on time, this will also allow us to find the qualitative dependence of power on time.

- $\mathbf{V = 0}$ **to** $\mathbf{V_0}$**.** In this part, the whole wire will be in the ground state, so that the power will be given by

$$P = \frac{V^2}{R} = \frac{V^2 A}{\rho_1 L} = P_0 \left(\frac{V}{V_0} \right)^2. \tag{12}$$

  At the end of this section, $V = V_0$, and hence $P = P_0$.

- $\mathbf{V = V_0}$ **to** $\mathbf{\sqrt{8/3} V_0}$**.** Over this section, the wire will undergo a phase transition. Since the current is approximately constant at I_t, the power, given by

$$P = I_t V, \tag{13}$$

  will increase linearly with voltage. The power will jump at the beginning of this section due to the sudden phase change to the value of

$$P = I_t V_0 = \sqrt{2/3} P_0, \tag{14}$$

  while at the end of this segment, we will have

$$P = \sqrt{2/3} P_0 \frac{\sqrt{8/3} V_0}{V_0} = \frac{4}{3} P_0, \tag{15}$$

  where I have used the fact that P is linear in V.

- $\mathbf{V = \sqrt{8/3} V_0}$ **to** $\mathbf{V_0 \sqrt{2}}$**.** Over this section, the wire will have constant resistivity $2\rho_1$, so that

$$P = \frac{V^2}{R} = \frac{V^2 A}{2\rho_1 L} = \frac{1}{2} P_0 \left(\frac{V}{V_0} \right)^2. \tag{16}$$

  At the end of this section, $V = V_0 \sqrt{2}$, so that $P = \frac{1}{2} P_0 \sqrt{2}^2 = P_0$.

- $\mathbf{V = V_0 \sqrt{2}}$ **to** $\mathbf{\sqrt{2/3} V_0}$**.** The wire will again undergo a phase transition, so that, for the same reason (that the current is constant at I_t), the power will be a linear function of V. Again, the power will jump at the beginning of this section, its new value being

$$P = I_t (V_0 \sqrt{2}) = \frac{2}{\sqrt{3}} P_0, \tag{17}$$

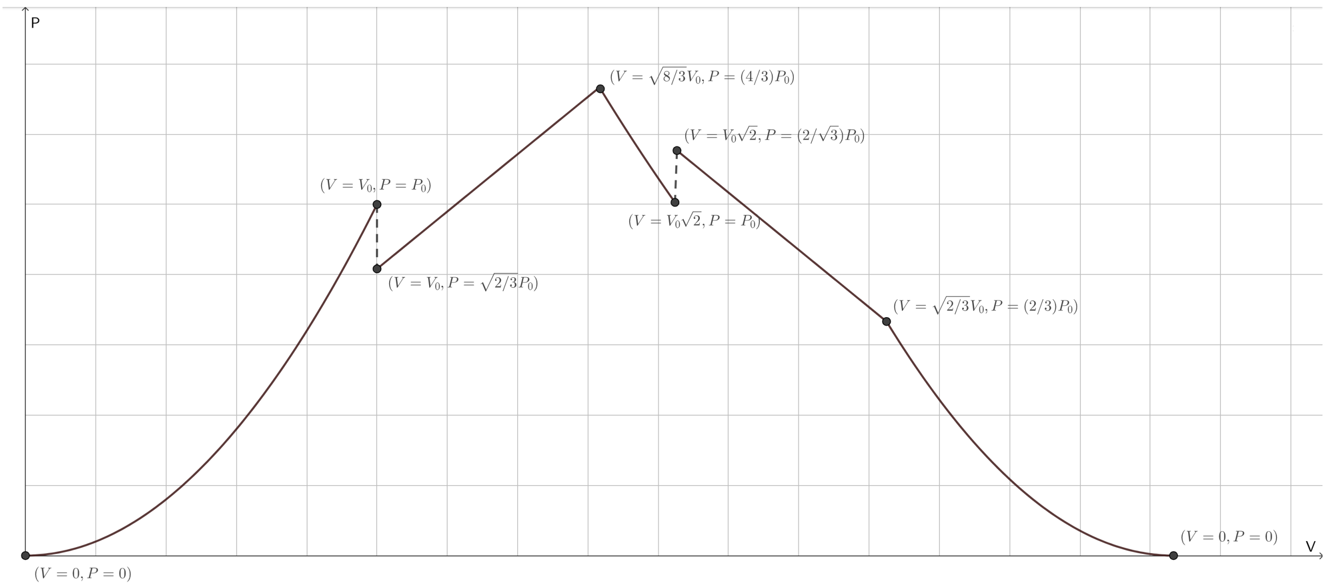
and at the end, we will have

$$P = \frac{2}{\sqrt{3}} P_0 \frac{\sqrt{2/3}V_0}{V_0\sqrt{2}} = \frac{2}{3} P_0. \tag{18}$$

- $\mathbf{V = \sqrt{2/3}V_0}$ **to 0.** Over this section, the resistivity of the wire is constant at ρ_1, so as in the first section,

$$P = P_0 \left(\frac{V}{V_0}\right)^2. \tag{19}$$

Using all this, we can now sketch the $P(t)$ graph. Since the voltage increases linearly with time, the shape of the functions $P(V)$ will be conserved in the transition to the $P(t)$ graph; for instance, the quadratic sections will be parabola sections, and the linear sections will be line segments. The phase transition ends at $V = \sqrt{8/3}V_0$, which means that this is the point where the voltage will start decreasing. The graph is shown on the next page.


P
$(V = \sqrt{8/3}V_0, P = (4/3)P_0)$
$(V = V_0\sqrt{2}, P = (2/\sqrt{3})P_0)$
$(V = V_0, P = P_0)$
$(V = V_0\sqrt{2}, P = P_0)$
$(V = V_0, P = \sqrt{2/3}P_0)$
$(V = \sqrt{2/3}V_0, P = (2/3)P_0)$
$(V = 0, P = 0)$
V
$(V = 0, P = 0)$