Physics Cup 2023, Problem 2

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1 Introduction

At first the temperature of the wire is almost constant along it but it will not remain so during the whole process. So, we introduce $T \equiv T(l,t)$ and $T(l,0) = T_0$ for all l; l being the coordinate of a point on the wire measured from its left end where we define the potential v to be 0: $v \equiv v(l,t)$, $v(0,t) = 0$ and $v(L,t) = V(t)$. We disregard the dimensions of the wire in directions orthogonal to l because $L \gg \sqrt{A}$. Also as the resistivity can take two values at any point along the wire and it can change at any given moment we define $\rho \equiv \rho(l,t)$ and $\rho(l,0) = \rho_1$.
Because the voltage $V(t)$ applied changes very slowly with time we neglect any effects of the inductance of the wire and take it to be electrically neutral throughout at all times. In essence we take the current I through the wire to be constant along it ($I(l) = I$) and equal to the value it would have in equilibrium at constant V and $\rho(l)$ in time.
We can differentiate between five "phases" in the behavior of the system. At first the whole wire has resistivity ρ_1 (Phase A). In the second phase part of the wire changes its phase so the resistivity is no longer constant along the wire (Phase B). In phase C the whole wire is in its second phase. In the third phase (D) part of the wire reverts to its initial phase. In the fourth phase (E) the whole wire has resistivity ρ_1 and lasts until the voltage drops to zero.

2 Phase A

We observe a very short segment of the wire dl which has a voltage dv between its two ends. Thus the current through it is

$$I = \frac{dv}{\rho \frac{dl}{A}} = \frac{dvA}{dl\rho_1} \tag{1}$$

we have

$$\int_0^L Idl = \int_0^V \frac{A}{\rho_1} dv \tag{2}$$

$$IL = \frac{A}{\rho_1} V \tag{3}$$

$$I(t) = \frac{V(t)A}{\rho_1 L} \tag{4}$$

The power of Joule heating in this short segment of wire, dP_J, is from eq. 1 and 4

$$dP_J = Idv = I\frac{dv}{dl}dl = \frac{I^2\rho_1}{A}dl = \frac{V^2 A}{\rho_1 L^2}dl \tag{5}$$

The heat loss to the environment per unit time is

$$dP_L = \alpha(T - T_0)dl \tag{6}$$

Since the change in voltage is very slow in relation to thermalization time, we may take

$$dP_L = dP_J \tag{7}$$

$$\alpha(T - T_0)dl = \frac{V^2 A}{\rho_1 L^2} dl \tag{8}$$

$$T = T_0 + \frac{V^2 A}{\alpha \rho_1 L^2} \equiv T_0 + \mathcal{T}(V) \tag{9}$$

So we have found the temperature of the wire (essentially constant along it for now) in relation to the voltage. This phase continues until $T = T_c$. The power emitted to the environment during this phase by the whole wire is

$$P_L = \int_0^L dP_L = \int_0^L \alpha(T - T_0)dl = \alpha L(T - T_0) \tag{10}$$

But also

$$P_L = \int_0^L dP_L = \int_0^L dP_J = \int_0^L \frac{V^2 A}{\rho_1 L^2} dl = \frac{V^2 A}{\rho_1 L}. \tag{11}$$

Thus $P_L \propto t^2$ because V grows linearly and at $T = T_c$ we have $P_L = \alpha L(T_c - T_0) = P_0$. From eq. 9 it is clear that the voltage at that point will be $V = L\sqrt{\alpha\rho_1(T_c - T_0)/A} = V_0$.

3 Phase B

Although in the last section we took all the variables to be essentially constant along the wire, since there are small variations in the cross section A and the "constant" α along the wire, there will be small variations in rates of Joule heating (eq. 5) and heat loss to the environment (eq. 6) and thus the temperature (eq. 9) along the wire. That means that some points will reach temperature T_c slightly before others and will therefore change phase first. These points become the kernels of expanding regions of wire which have resistivity $\rho_2 = 2\rho_1$. Let's denote with ξ the total length of wire which is in the second phase. The total resistance of the wire may be calculated as

$$R = \rho_2 \frac{\xi}{A} + \rho_1 \frac{L - \xi}{A} = \frac{\rho_1}{A}(2\xi + L - \xi) = \frac{\rho_1 L}{A}\left(1 + \frac{\xi}{L}\right) \tag{12}$$

The current flowing through the wire must then be

$$I = \frac{V}{R} = \frac{VA}{\rho_1 L}\left(1 + \frac{\xi}{L}\right)^{-1} \tag{13}$$

To calculate the power of Joule heating of a short segment of wire (which can be taken to be in one phase in its entirety) we first calculate (from eq. 1 and 13)

$$\frac{dv}{dl} = \frac{I\rho}{A} = \frac{\rho V}{\rho_1 L}\left(1 + \frac{\xi}{L}\right)^{-1} \tag{14}$$

$$\left(\frac{dv}{dl}\right)_1 = \frac{I\rho}{A} = \frac{V}{L}\left(1 + \frac{\xi}{L}\right)^{-1} \tag{15}$$

$$\left(\frac{dv}{dl}\right)_2 = \frac{I\rho}{A} = \frac{2V}{L}\left(1 + \frac{\xi}{L}\right)^{-1} \tag{16}$$

where we denote with subscript 1 the phase with $\rho = \rho_1$ and with subscript 2 the phase with $\rho = \rho_2$.
The Joule heating:

$$dP_J = Idv = \frac{VA}{\rho_1 L}\left(1 + \frac{\xi}{L}\right)^{-1} \cdot \frac{\rho V}{\rho_1 L}\left(1 + \frac{\xi}{L}\right)^{-1} dl \tag{17}$$

$$dP_J = \frac{\rho V^2 A}{\rho_1^2 L^2}\left(1 + \frac{\xi}{L}\right)^{-2} dl \tag{18}$$

$$(dP_J)_1 = \frac{V^2 A}{\rho_1 L^2}\left(1 + \frac{\xi}{L}\right)^{-2} dl \tag{19}$$

$$(dP_J)_2 = \frac{2V^2 A}{\rho_1 L^2}\left(1 + \frac{\xi}{L}\right)^{-2} dl \tag{20}$$

Again we use the fact that thermalization is really quick which also means that power spent on it is really small. So at almost all points we have

$$dP_J = dP_L \tag{21}$$

$$\frac{\rho V^2 A}{\rho_1^2 L^2}\left(1+\frac{\xi}{L}\right)^{-2} dl = \alpha(T-T_0)dl \tag{22}$$

$$T = T_0 + \frac{\rho V^2 A}{\alpha \rho_1^2 L^2}\left(1+\frac{\xi}{L}\right)^{-2} = T_0 + \frac{\rho}{\rho_1}\mathcal{T}(V)\left(1+\frac{\xi}{L}\right)^{-2} \tag{23}$$

$$T_1 = T_0 + \mathcal{T}(V)\left(1+\frac{\xi}{L}\right)^{-2} \tag{24}$$

$$T_2 = T_0 + 2\mathcal{T}(V)\left(1+\frac{\xi}{L}\right)^{-2} \tag{25}$$

Now, at $V = V_0$ and while $\xi/L \ll 1$ because $\mathcal{T}(V_0) = T_c - T_0$ we have $T_1 = T_c$ and $T_2 = T_0 + 2(T_c - T_0)$. Meaning that at first most of the wire stays at T_c but the points which undergo the phase transition first heat up rapidly to $2T_c - T_0$. However at points which border the regions of wire whose phase differs there appears a very large temperature gradient and inevitably some heat flow develops from the regions of phase 2 to phase 1. But since it is stated that this heat flow along the wire can be almost everywhere neglected we consider it to only be appreciable at the stated points and that it doesn't have an impact on the temperature distribution elsewhere along the wire, i.e. the regions in phase 2 have essentially the same temperature T_2 throughout and the regions in phase 1 the same temperature T_1 throughout.
Although we approximate that the heat flux along the wire is negligible except at the points bordering the regions of different resistivity to facilitate a simpler function of temperature along the wire (which makes the calculations of heat loss to the environment much easier), the temperature cannot in reality have discrete jumps between the "equilibrium" temperatures of the two phases. In reality, for a certain voltage (taken to be constant), a quasiequilibrium forms such that there is a very sharp but continuous temperature transition between the two phases.
Now let's investigate one such point bordering two neighboring regions of different phases. Without loss of generality let the region of resistivity ρ_1 be to the left of the origin ($x < 0$) of a coordinate system whose x-axis is aligned with the wire and let the region of resistivity ρ_2 be to the right of the origin ($x > 0$). Relatively speaking, far to the right of the origin the temperature must tend to T_2, the equilibrium temperature of the second phase, and far to the left it tends to T_1, the equilibrium temperature of the first phase. At the origin the temperature is T_c almost by definition because if it were higher then there would be points of phase two to the left of the origin or points in phase one to the right of it. This follows because T is obviously a continuous monotonically increasing function (since $T_1 < T_c < T_2$).
We write down the equation of net power influx for a short segment of the wire, dx, while taking into account heat flux along the wire. Initially we presume that a quasiequilibrium is reached such that $T(l)$ is constant in time so that the net influx of power must be zero.

$$0 = \frac{nV^2 A}{\rho_1 (L+\xi)^2}dx - \alpha(T-T_0)dx + \left(kA\frac{dT}{dx}\bigg|_{x+dx} - kA\frac{dT}{dx}\bigg|_{x}\right) \tag{26}$$

Where the first term corresponds to Joule heating internal to the segment, the second term to heat loss to the environment and the third and fourth terms to heat flux into the segment from parts of the wire to the right and to the left of the segment respectively. The thermal conductivity is k and we designate if the segment is in phase 1 or 2 with n ($n = \frac{\rho}{\rho_1}$). This equation is equivalent to

$$\begin{aligned}
0 &= \frac{nV^2 A}{\rho_1 (L+\xi)^2} - \alpha(T-T_0) + kA\frac{d^2T}{dx^2} \\
0 &= kAT'' - \alpha T + \left(\alpha T_0 + \frac{nV^2 A}{\rho_1 (L+\xi)^2}\right) \\
0 &= \frac{1}{\omega^2}T'' - T + T_n
\end{aligned} \tag{27}$$

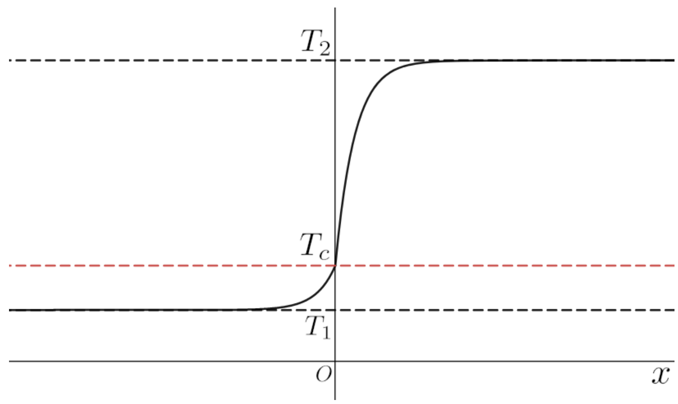

Figure 1: Temperature distribution around a border point

where $\omega = \sqrt{\frac{\alpha}{kA}}$. This is a standard second-order linear ODE which can be easily solved for the initial conditions

$$\lim_{x \to -\infty} T = T_1 \tag{28}$$
$$T(0) = T_c \tag{29}$$

for $n = 1$ and

$$\lim_{x \to +\infty} T = T_2 \tag{30}$$
$$T(0) = T_c \tag{31}$$

for $n = 2$. The solution is

$$T = (T_c - T_1)e^{\omega x} + T_1 \tag{32}$$

for $x \leq 0$ ($n = 1$) and

$$T = (T_c - T_2)e^{-\omega x} + T_2 \tag{33}$$

for $x \geq 0$ ($n = 2$).

From this we can calculate the derivative of the temperature at $x = 0$:

$$(T'(0))_1 = (T_c - T_1)\omega \tag{34}$$
$$(T'(0))_2 = (T_2 - T_c)\omega \tag{35}$$

In general this means that T' is discontinuous at 0. But that would be in contradiction with the assumption of equilibrium. If there is a discontinuity in T' then T'' diverges at that point, meaning that there is certainly net heat flow to (or from) this point[1] as heat influx is proportional to T''. So equilibrium can only be achieved if

$$T_c - T_1 = T_2 - T_c \tag{36}$$

holds. Moreover, this equilibrium is stable.
If we have $T_2 - T_c > T_c - T_1$ then for a short segment of wire containing $x = 0$, dx, we can write down the expression for net power influx for this segment:

$$\frac{\overline{n}V^2 A}{\rho_1 (L + \xi)^2} dx - \alpha(T_c - T_0)dx + kA\left((T'(0))_2 - (T'(0))_1\right) = kA\left((T'(0))_2 - (T'(0))_1\right) > 0 \tag{37}$$

There is a finite positive influx of power into this infinitesimal segment of wire[2]. This means that this segment is going to change to phase 2 and then continue to heat up as the new point bordering the two phases "moves" to the left of the origin. The fraction of wire in phase 2 increases (ξ increases) thereby decreasing T_2 and T_1 (eq.24 and 25) but T_2 decreases at faster rate with the increase of ξ than T_1 so as the border point continues to "move" to the left, eventually we'll have $T_2 - T_c = T_c - T_1$ and equilibrium will be reached. The opposite but essentially the same

[1] More precisely, from the infinitesimal segment of wire containing this point
[2] This presents no issue because this power influx will only be present for an infinitesimal amount of time

happens if initially $T_2 - T_c < T_c - T_1$.
The rate at which the border point moves depends on the thermal conductivity of the wire as well as the latent heat of the phase transition but the assumption is that thermalization happens very fast in comparison to the rate of change of the voltage so we take it to be essentially instantaneous.
Now let's calculate the total power emitted to the environment at any point in time:

$$\begin{aligned}
P_L &= \int_{wire} dP_L \\
&= \int_{wire} dP_J = \int_1 dP_J + \int_2 dP_J \\
&= \int_1 \frac{V^2 A}{\rho_1 L^2}\left(1+\frac{\xi}{L}\right)^{-2} dl + \int_2 \frac{2V^2 A}{\rho_1 L^2}\left(1+\frac{\xi}{L}\right)^{-2} dl \\
&= \frac{V^2 A}{\rho_1 L^2}\left(1+\frac{\xi}{L}\right)^{-2}\left(\int_1 dl + 2\int_2 dl\right) \\
&= \frac{V^2 A}{\rho_1 L^2}\left(1+\frac{\xi}{L}\right)^{-2}\left(\int_{wire} dl + \int_2 dl\right) \\
&= \frac{V^2 A}{\rho_1 L^2}\left(1+\frac{\xi}{L}\right)^{-2}(L+\xi) \\
&= \frac{V^2 A}{\rho_1 (L+\xi)}
\end{aligned} \tag{38}$$

From the condition of eq.36 we calculate plugging in eq.24 and 25

$$L + \xi = L\sqrt{\frac{3\mathcal{T}(V)}{2(T_c - T_0)}} = \sqrt{\frac{3}{2}}\frac{LV}{V_0} \tag{39}$$

Then from eq.38 and eq.39

$$P_L = \sqrt{\frac{2}{3}}\frac{VV_0 A}{\rho_1 L} \tag{40}$$

At the begining of phase B when $V = V_0$ from eq.39 and 40 we have $L+\xi = \sqrt{3/2}L$ and $P_L = \sqrt{2/3}P_0$. Afterwards, P_L increases linearly with V and therefore with time according to eq.40. This continues until ξ increases to $\xi = L$ at $V = \sqrt{8/3}V_0$ (eq.39). At that point $P_L = 4/3P_0$.

4 Phase C

The whole wire is now in phase 2 and has temperature $T_2 = T_0 + \frac{4}{3}\mathcal{T}(V_0)$ which is larger than T_c. From eq.38 we know that the emitted power equals ($\xi = L$)

$$P_L = \frac{V^2 A}{2\rho_1 L} \tag{41}$$

as long as the entire wire is in phase 2. Therefore, emitted power starts to decrease and its dependence on time represented in the graph will be a convex curve. T_2 gradually decreases until it is equal to $T_2 = T_c$ which happens when $V = \sqrt{2}V_0$ (from eq.25). At this point $P_L = P_0$.

5 Phase D

Again, because of small variations in A and α, some points will reach T_c slightly before others, will change back to phase 1 and quickly cool down to T_1. The same equilibrium asserts itself as in phase B as we have parts of wire in phase 1 and parts in phase 2. Meaning that eq.39 and eq.40 hold again. From eq.39 we have that $L + \xi = \sqrt{3}L$ and from eq.40 we have $P_L = 2/\sqrt{3}P_0$ at the start of this phase (when $V = \sqrt{2}V_0$). As the voltage continues to decrease, the emitted power continues to decrease linearly (eq.40) and ξ decreases steadily as well until the entire wire has changed back to phase 1 (when $\xi = 0$). This happens when $V = \sqrt{2/3}V_0$ according to eq.39 and the power emitted to the environment at that point is $P_L = 2/3P_0$ according to eq.40.

6 Phase E

The wire, being in its entirety in phase 1, behaves again according to eq. 11. At the start of this phase we have $P_L = 2/3P_0$ from plugging $V = \sqrt{2/3}V_0$ in eq.11. Also, it is clear that P_L will decrease convexly with voltage and time until it is zero.

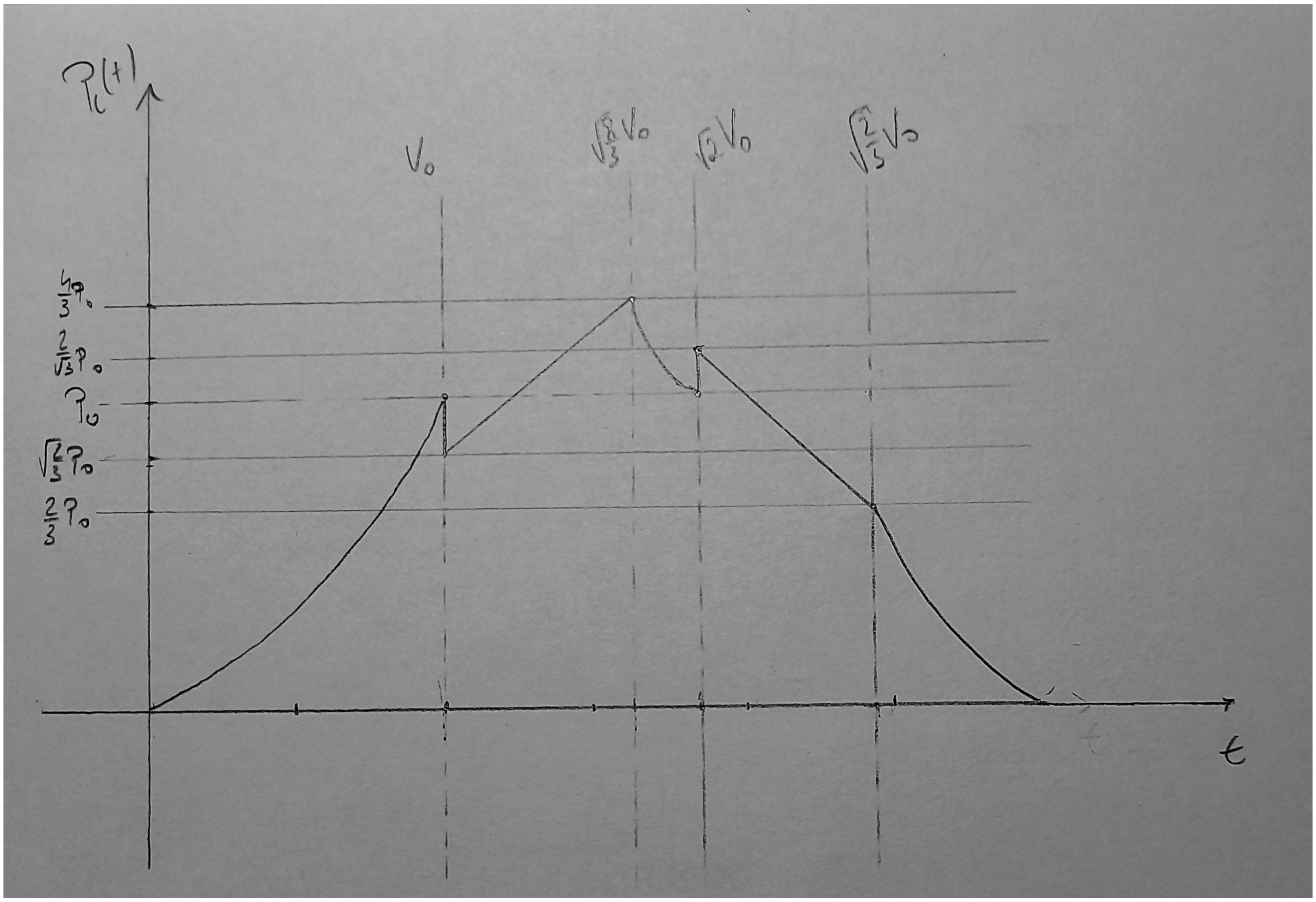

Figure 2: Power emitted to the environment vs. time