Dobrica Jovanovic - Physics Cup 2023 Problem 3.

- Answer: attractive force of amplitude $\boxed{F \approx \frac{\mu_0 m^2}{R^5} L}$

- Proof:

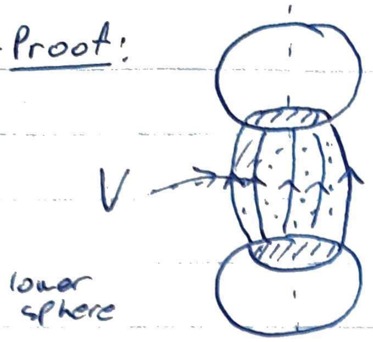

Fig 1

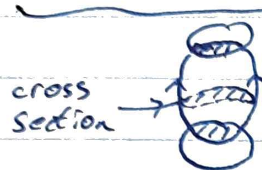

Fig 3.

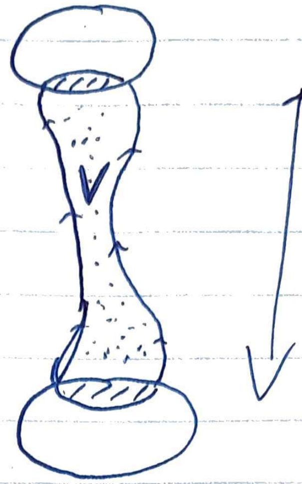

Fig 2

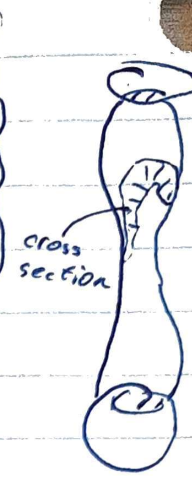

Fig 4

- Focus on the magnetic field coming from the lower sphere, precisely from a some "circle" (~~rotationally~~ centered at the main axis). Let's call the volume between the field lines ~~of th~~ coming from this circle V. Let's analyze how do field lines from this circle ~~and~~ interplay with V after we seperate the spheres as in Fig 2. Well, due to rotational symmetry, we should have both rotationaly symmetric field lines, and space location of V. Let's take some cross-section of the fluid as in Fig 1. Let the same cross-secton look something like in Fig 4 for the 2nd case. Due to superconductivity of the fluid, the magnetic flux ($\propto$ # of field lines) on the cross-section should be same in both cases. This implies that the same field lines that enter the cross section ~~as~~ in the 1st case, should enter the cross-section in 2nd case.

~~Indeed, due to rotational symmetry we can assume that~~

This further implies that for any field lines coming from some small area S_1 from the lower sphere (Fig 5), we expect that the enveloping volume of fluid should exactly stay between these lines, even in the 2nd case too (Fig 6).

Fig 5

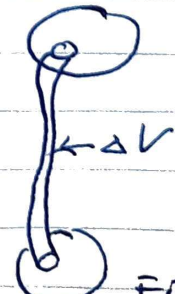

Fig 6

Now, fluid is incompressible, so the volumes in Fig 5 and Fig 6 are same. Say that $\tilde{S}_1$ is average cross sectional area in the first case for Fig 5 tube, and $\tilde{S}_2$ is average cross sectional area for Fig 6 tube

Due to conservation of volume, we can ~~som~~ approximate that $\tilde{S_1} \cdot$ (length of tube in Fig 5) $= \tilde{S_2} \cdot$ (length of tube in Fig 6), We can further approximate that these lengths are the lengthscales of the problem, so that $\boxed{\tilde{S_1} \cdot R \approx \tilde{S_2} \cdot L}$ (1).

As we said, we have the same ~~t~~ magnetic flux in both cases, so if we say that $\tilde{B_1}$ is an average magnetic field strength in the tube in Fig 5, and $\tilde{B_2}$ similar for tube in Fig 6 we have:

$$\boxed{\tilde{B_1} \cdot \tilde{S_1} = \tilde{B_2} \cdot \tilde{S_2}} \quad (2)$$

(1),(2) together give $\boxed{\frac{\tilde{B_2}}{\tilde{B_1}} \approx \frac{L}{R}}$ (3)

Now ration $\frac{L}{R}$ is independent of choice of the starting tube, so we could approximate that we get this result for all tubes.

This implies that the energy density of magnetic field in 2nd case should be $\boxed{\sim \left(\frac{L}{R}\right)^2}$ higher than in the 1st case.

Now, we can approximate magnetic field energy in the first case as $W_1 \approx \frac{\mu_0 m^2}{R^3}$ (dimensional analysis gives this, given that m and R are only important physical quantities in the 1st case). We could also resort to some approximative calculations, which would give us the same answer.

$$\Rightarrow W_2 \approx \frac{\mu_0 m^2}{R^3} \cdot \frac{L^2}{R^2} \Rightarrow F_2 = -\frac{\partial W_2}{\partial L} \hat{L} \approx -\frac{\mu_0 m^2}{R^5} \cdot L .$$