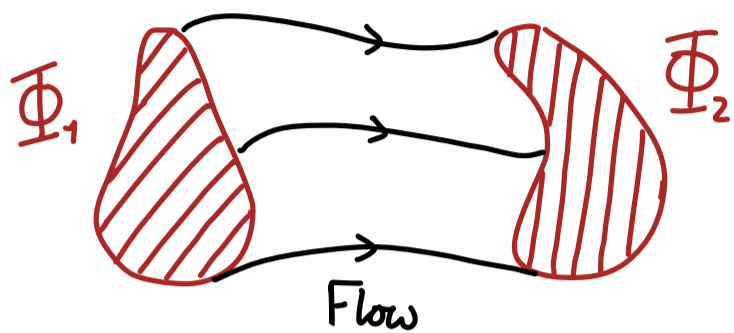


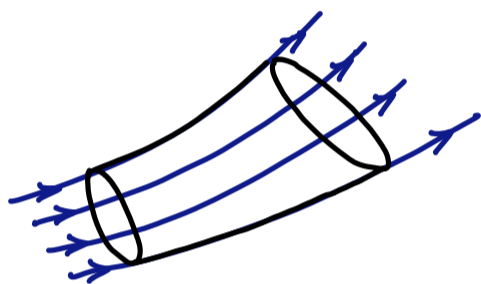
This is a problem in magnetohydrodynamics (MHD).

The key idea is 'flux freezing' (or Alfvén's Theorem): consider any loop moving with the fluid flow. The EMF around this loop must always remain zero because the fluid material is superconducting. So, the magnetic flux through the loop cannot change.



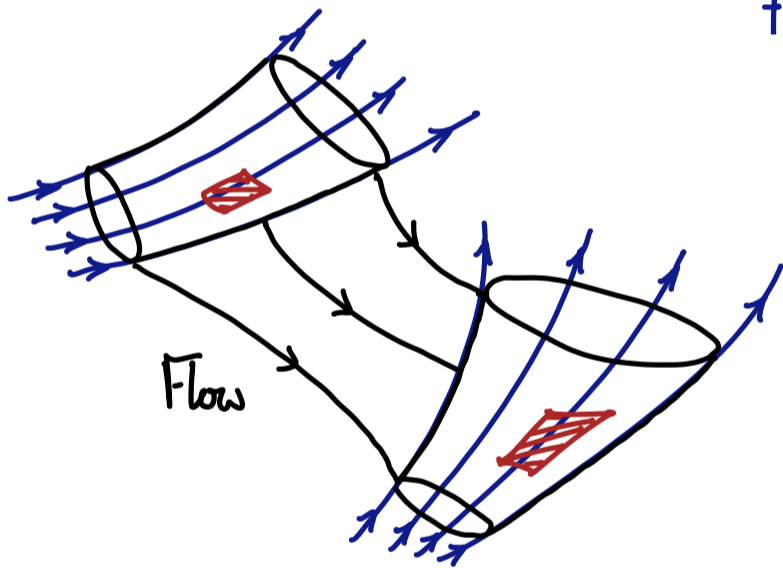
This loop moves with the flow. The magnetic flux through it doesn't change, $\Phi_1 = \Phi_2$.

Now consider a flux tube, meaning a closed surface in the shape of a tube with zero magnetic flux through any part of its sides (so it's like a bundle of field lines):



Blue: magnetic field lines
Black: flux tube surface

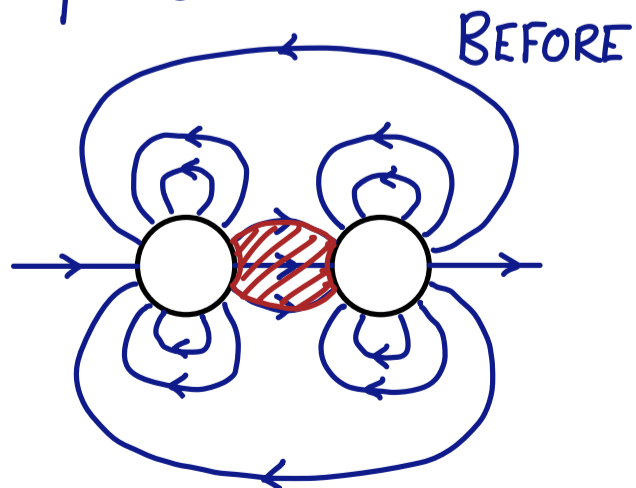
Let this imaginary, closed surface move with the flow.



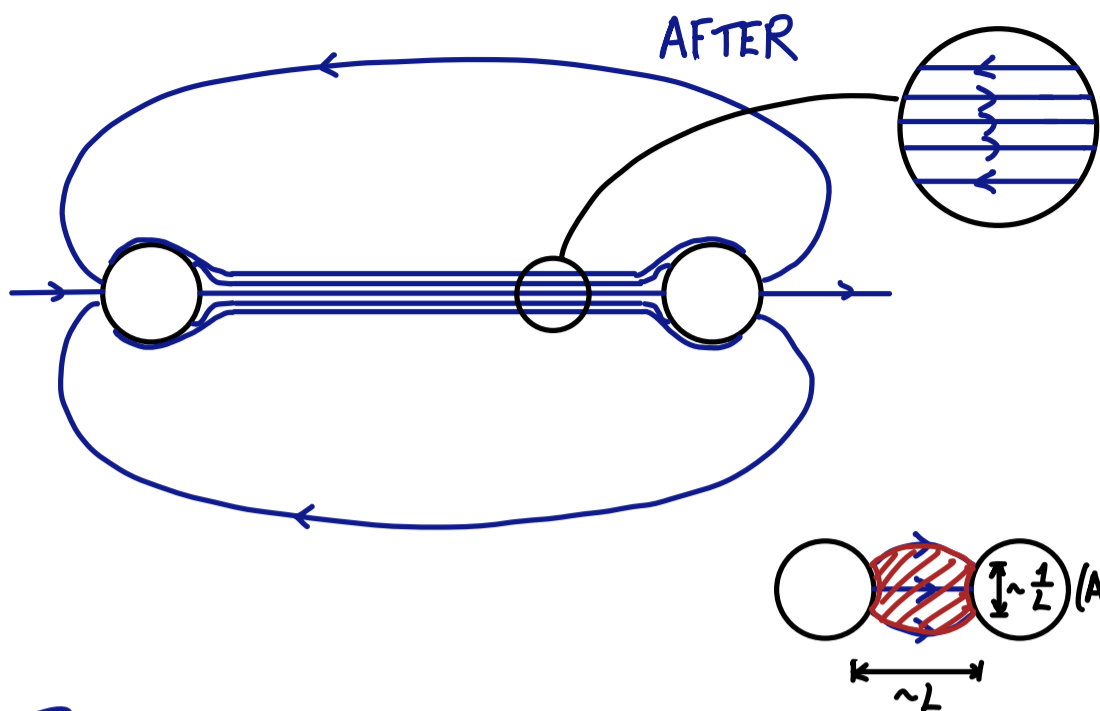
Claim: At later times, the magnetic flux through the sides of the tube is still zero.

Proof: Consider the red patch on the surface. The flux through it is initially zero and cannot change (Alfvén's Theorem). So, the flux through it remains zero. This is true for any part of the side of the tube. \square

Note also that the flux through the ends of the tube doesn't change as it moves with the flow. Finally, note that the volume of the flux tube doesn't change if the flow is incompressible.



In this problem, as a flux tube (shown in red) moves with the flow it must get very long and therefore very thin (to conserve its volume). Thus the field lines are stretched out to form a very narrow flux tube (see below).



There is a force because the magnetic energy is much greater now. The energy density is $B^2/2\mu_0$ which is large because the field lines are close together.

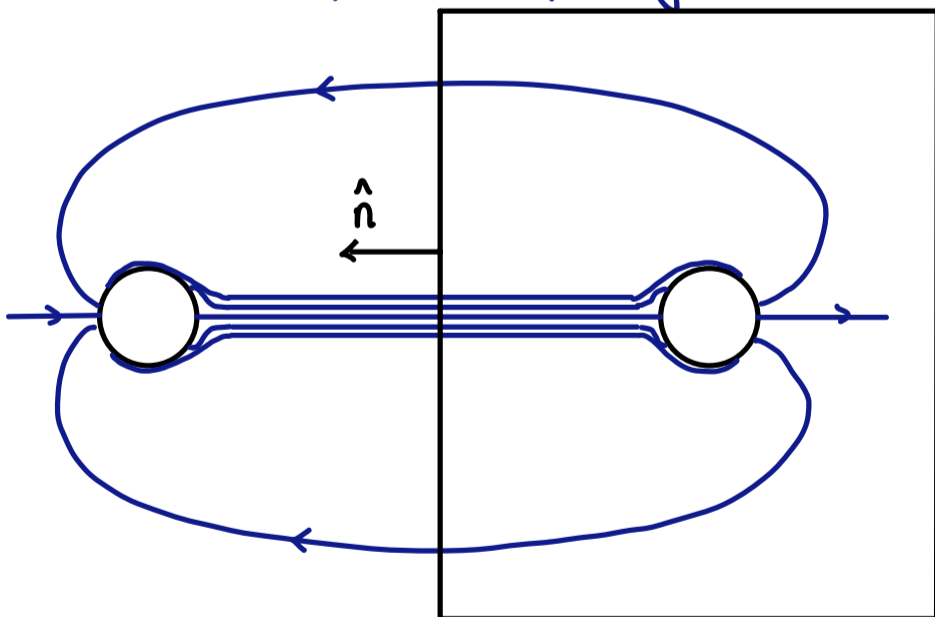
The cross-sectional area of each axisymmetric flux tube scales with $1/L$, to preserve volume.

To conserve the magnetic flux through the tube, the magnetic field in it must scale with L .

Then the magnetic energy in the tube scales with L^2 . So, the energy of the system scales with L^2 , which means the work done if L is increased by δL is proportional to $(L+\delta L)^2 - L^2 \approx 2L\delta L$. The force is therefore proportional to L (similar to flux tubes connecting quarks in QCD!).

By dimensional analysis, the force is $\sim \frac{\mu_0 m^2 L}{R^5}$ (attraction).

This argument ignores thermal energy, which might be important because the pressure in the fluid changes to maintain incompressibility. For example, the pressure outside the narrow flux tube must be much larger than the pressure inside, because magnetic forces try to widen the tube. So I prefer the following method.



Draw a large box around one dipole as shown. Since the fluid is assumed to be in equilibrium, the force required to hold the dipole in place balances the force exerted by the fluid and field on the contents of this box.

This force is the integral of the stress over the surface of the box. The stress tensor is:

$$T = \underbrace{\frac{\vec{B}\vec{B}}{\mu_0} - \frac{B^2}{2\mu_0}\mathbf{I}}_{\text{Maxwell stress}} - \underbrace{(P - P_\infty)\mathbf{I}}_{\text{Pressure relative to pressure at infinity } P_\infty}$$

Unit matrix/dyad

(assuming ideal MHD, so isotropic pressure, no dissipation or viscosity)

Since B decays like $1/r^3$, only the part of the box on the plane of symmetry contributes.

$$\vec{F} = \int_{\text{box}} \mathbf{T} \cdot \hat{n} \, dS = \int \frac{B^2}{\mu_0} \hat{n} - \underbrace{\left(\frac{B^2}{2\mu_0} + (P - P_\infty) \right)}_{\text{Magnetic plus thermal pressure}} \hat{n} \, dS$$

Magnetic plus thermal pressure

The force density on the fluid at the plane of symmetry is

$$\vec{f} = \underbrace{\frac{\vec{B} \cdot \nabla \vec{B}}{\mu_0}} - \nabla \left(\frac{B^2}{2\mu_0} + (P - P_\infty) \right) = 0$$

This term is negligible because field lines are basically straight. It is $\sim B^2/\mu_0 L$, while the pressure gradients are $\sim B^2/\mu_0 w$, where w is the width of the flux tube.

So, $\frac{B^2}{2\mu_0} + (P - P_\infty) = \text{const.}$ on the plane of symmetry. By going far away, so $B \rightarrow 0$ and

$P \rightarrow P_\infty$, so this constant is zero.

$$\text{Thus } \vec{F} = \int_{\text{plane of symmetry}} \frac{B^2}{\mu_0} \, dS \, \hat{n}.$$

Now the previous arguments for how B scales apply. Note that the thermal pressure doesn't matter after all.