# Physics Cup 2023, Problem 3

Luka Passek-Kumerički

Because of infinite conductivity of the fluid, the magnetic flux through any loop co-moving with the fluid must be constant. If it were not so, a voltage would be induced in this loop by a change in flux according to Faraday's law. But this would result in infinite currents. By dimensional analysis, the total energy contained in the magnetic field initially must be

$$U_0 \propto \mu_0 \frac{m^2}{R^3} \tag{1}$$

Let's imagine a very thin co-moving cylindrical volume coaxial with the line connecting the centers of the spheres each base of which touches the cylinders. As the fluid and its velocity field are continuous the volume of this cylinder (although it may not strictly remain a cylinder) stays connected (same as for the co-moving loops) and its volume stays constant because of the incompressibility condition. Furthermore, the "bases" of this co-moving volume will at all times remain at the surfaces of the spheres. This follows from the no-penetration condition on the velocity field at the surface of the spheres. In the reference frame of a sphere, the fluid velocity must be tangential to the sphere, meaning that co-moving points at the sphere surface always remain there.
As the spheres are moved apart, this cylinder gets stretched and thinned out because its bases are stuck to the spheres but its volume must stays constant. (Here the condition that the magnetic dipole moments of the spheres are parallel to the line connecting their centers is important because it gives the problem cylindrical symmetry which is why we can say that the cylinder remains coaxial at all times.) The reduction of the area of the cross-section of this volume perpendicular to the axis of cylindrical symmetry (and its elongation) implies an increase in the magnetic field along the axis because the flux through such co-moving cross-sections remains constant. It is obvious that the average area of such cross-sections must be inversely proportional to the distance between the spheres but it also seems implausible that there be large variation in the area along the axis because that would imply large variation in the magnetic field component parallel to the axis which seems energetically unfavorable. So we take the area $A$ roughly constant along the axis from which we have, because of constant volume, that it is inversely proportional to the distance between the spheres $l$ for all such cross-sections:

$$A \propto \frac{1}{l} \tag{2}$$

From constant flux we have that the magnetic field component parallel to the axis inside the cylinder is inversely proportional to said area:

$$B_{||} \propto \frac{1}{A} \propto l \tag{3}$$

from which we have

$$B_{||} = B_{||,0} \frac{L}{R} \tag{4}$$

after the spheres have been moved apart where subscript 0 denotes the initial state.
The energy contained in the field within the cylinder is proportional to the square of the magnetic field. We may ignore the radial component of the magnetic field when calculating the field energy within the cylinder after the spheres have been moved apart because it is actually decreasing because of the stretching, so

$$U_c \propto U_{c,0} \frac{L^2}{R^2} \tag{5}$$

where subscript $c$ denotes "inside of the cylinder".
Let's now imagine a co-moving coaxial cylindrical volume that is just large enough to envelop both of the spheres. So, even as they are moved apart the volume must always contain them. In this case, the cross-sectional area of the volume will not remain constant along it because the two ends of it must envelop the spheres of fixed radius. But between the spheres we still presume that, as the average cross-sectional area is inversely proportional to $l$, so are all of the possible cross-sections roughly. The rest of the argument from above then follows equivalently. The

only difference is that the magnetic field energy contained in the initial cylinder is a significant portion of the total energy as most of the field energy is contained close to the spheres, because the magnetic field strength from dipoles falls of as $1/r^3$. The increase in the total energy is then almost exclusively due to this stretching-of-the-field effect between the spheres, because after the spheres are moved apart to distance $L \gg R$, the parts of the magnetic field which weren't stretched have negligible energy with respect to the stretched region.
We then have that the total energy at the end is

$$U \propto U_0 \frac{L^2}{R^2} = \mu_0 \frac{m^2 L^2}{R^5} \tag{6}$$

From the problem's cylindrical symmetry the force which we must then exert to hold the spheres in place is parallel to the axis and

$$F = -\frac{\mathrm{d}U}{\mathrm{d}L} \tag{7}$$

$$\Rightarrow F \propto \mu_0 \frac{m^2 L}{R^5} \tag{8}$$