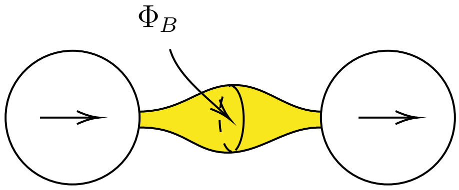
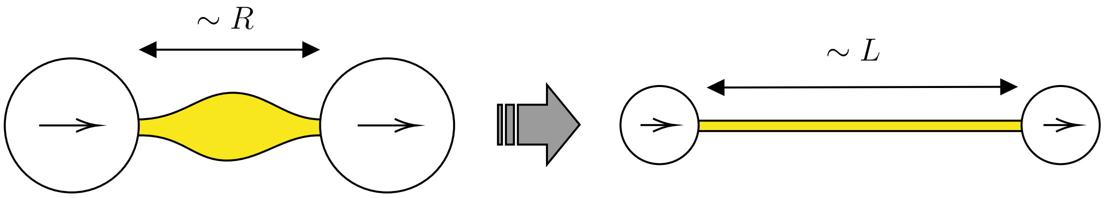
Physics Cup 2023 Problem 3



Consider a flux tube connecting the two spherical magnets. Its surface is parallel to the magnetic field lines, so the flux across any cross-section of the flux tube is a constant Φ_B. Because of the symmetry of the system with respect to the line joining the centres of the spheres together, the flux tube is a solid of revolution about that line. We only care about the flux tubes that connect the two magnets because they are the ones that change the most significantly during the movement of the spheres and contribute to the force F.



At the initial state, the flux tubes are of the length $\sim R$. As the spheres are separated, they are stretched to a new length which is $\sim L$ (ignore the size of the spheres as $L \gg R$ However, as the particles in the fluid must move along the field lines (i.e. move parallel to them), and the liquid is incompressible, the total mass enclosed in a flux tube must be constant, so its total volume should not change. Hence, any cross-sectional area is multiplied by a factor of $\sim R/L$. By the constant magnetic flux, we have:

$$(A)B_i = \Phi_B \approx \left(A \cdot \frac{R}{L}\right) B_f \quad \Rightarrow \quad B_f \sim B_i \left(\frac{L}{R}\right)$$

Then, consider the energy inside the flux tubes. The volume of the tubes haven't changed, but the magnetic field has increased by a factor. Since $E \propto B^2$, so the final energy is given by

$$E \sim E_0 \cdot \left(\frac{L}{R}\right)^2$$

Where E_0 is the initial energy in the gap between the two spheres. But what is the initial energy? We can estimate it by considering that the intial magnetic field in the gap is of the order $\mu_0 m/R^3$ and the initial volume is $\sim R^3$.

$$E_0 \approx \frac{B^2 V}{2\mu_0} \sim \frac{\mu_0^2 m}{R^3} \quad \Rightarrow \quad E \sim \frac{\mu_0 m^2 L^2}{R^5}$$

As the increased in the stored magnetic field energy is the major contribution to the force, we can write $F = -\mathrm{d}E/\mathrm{d}L$ to estimate the magnetic force magnitude to be approximately:

$$\boxed{F \sim \frac{\mu_0 m^2 L}{R^5}}$$